

Lifting properties of maximal lattice-free polyhedra

Gennadiy Averkov* and Amitabh Basu†

January 20, 2015

Abstract

We study the uniqueness of minimal liftings of cut-generating functions obtained from maximal lattice-free polyhedra. We prove a basic invariance property of unique minimal liftings for general maximal lattice-free polyhedra. This generalizes a previous result by Basu, Cornuéjols and Köppe [BCK12] for *simplicial* maximal lattice-free polytopes, thus completely settling this fundamental question about lifting for maximal lattice-free polyhedra. We further give a very general iterative construction to get maximal lattice-free polyhedra with the unique-lifting property in arbitrary dimensions. This single construction not only obtains all previously known polyhedra with the unique-lifting property, but goes further and vastly expands the known list of such polyhedra. Finally, we extend characterizations from [BCK12] about lifting with respect to maximal lattice-free simplices to more general polytopes. These nontrivial generalizations rely on a number of results from discrete geometry, including the Venkov-Alexandrov-McMullen theorem on translative tilings and characterizations of zonotopes in terms of central symmetry of their faces.

1 Introduction

Mixed-integer corner polyhedra. The mixed-integer *corner polyhedron* is the convex hull of a *mixed-integer set* of the following form:

$$X_f(R, P) := \left\{ (s, y) \in \mathbb{R}_+^k \times \mathbb{Z}_+^\ell : f + Rs + Py \in \mathbb{Z}^n \right\}, \quad (1.1)$$

where $k, \ell \in \mathbb{Z}_+, n \in \mathbb{N}$, $R \in \mathbb{R}^{n \times k}$, $P \in \mathbb{R}^{n \times \ell}$ and $f \in \mathbb{R}^n \setminus \mathbb{Z}^n$. The set $X_f(R, P)$ was first studied by Gomory [Gom69] for the purposes of generating cutting planes for general mixed-integer linear problems (MILPs). A short description of Gomory's idea is as follows. Consider a general MILP with a feasible region given in the standard form:

$$\left\{ x \in \mathbb{R}_+^h \times \mathbb{Z}_+^q : Ax = b \right\}. \quad (1.2)$$

Here $Ax = b$ is the defining linear system, h is the number of continuous variables and q is the number of integer variables. The simplex method applied to the linear relaxation of the MILP decomposes the variables of x into basic and non-basic ones. As a result, we can split

*Institute of Mathematical Optimization, Faculty of Mathematics, University of Magdeburg, Germany

†Department of Applied Mathematics and Statistics, The Johns Hopkins University, MD, USA

x into four vectors: the vector $s \in \mathbb{R}_+^k$ of non-basic continuous variables, the vector $t \in \mathbb{R}_+^m$ of basic continuous variables, the vector $y \in \mathbb{Z}_+^\ell$ of non-basic integer variables and the vector $z \in \mathbb{Z}_+^n$ of basic integer variables.

The basic variables can be expressed through the non-basic ones. That is, one has $z = f + Rs + Py$ and $t = g + Us + Vy$ for appropriate matrices U, V, R, P and appropriate vectors $f \in \mathbb{R}^n, g \in \mathbb{R}^m$. Gomory suggests relaxing the MILP by discarding the nonnegativity conditions on basic variables, that is, the conditions $t \in \mathbb{R}_+^m$ and $z \in \mathbb{Z}_+^n$ are relaxed to $t \in \mathbb{R}^m$ and $z \in \mathbb{Z}^n$. After this, z and t can be eliminated from the problem description, since they are expressed through the non-basic variables; the condition $z \in \mathbb{Z}^n$ can also be reformulated without any use of z as $f + Rs + Ry \in \mathbb{Z}^n$. This gives rise to the mixed-integer set $X_f(R, P)$ as defined in (1.1). Previous studies show that the corner polyhedron (the convex hull of $X_f(R, P)$) has a quite special facial structure, in sharp contrast to the facial structure of the convex hull of (1.2), which has much less structure in general. There has been a vast amount of literature, specially in the last 5-6 years, on utilizing the corner polyhedron for developing general-purpose solution methods in mixed-integer linear programming. We refer the reader to the survey [CCZ11a] for this line of research.

Cut generating functions. In the 1970s Gomory and Johnson [GJ72a, GJ72b, Joh74] suggested the following approach for generation of cuts for $X_f(R, P)$. We denote the columns of matrices R and P by r_1, \dots, r_k and p_1, \dots, p_ℓ , respectively. We allow the possibility that $k = 0$ or $\ell = 0$ (but not both). Given $n \in \mathbb{N}$ and $f \in \mathbb{R}^n \setminus \mathbb{Z}^n$, a *cut-generating pair* (ψ, π) for f is a pair of functions $\psi, \pi : \mathbb{R}^n \rightarrow \mathbb{R}$ such that

$$\sum_{i=1}^k \psi(r_i) s_i + \sum_{j=1}^\ell \pi(p_j) y_j \geq 1 \quad (1.3)$$

is a valid inequality (also called a *cut*) for the set $X_f(R, P)$ for every choice of $k, \ell \in \mathbb{Z}_+$ and for all matrices $R \in \mathbb{R}^{n \times k}$ and $P \in \mathbb{R}^{n \times \ell}$. We emphasize that cut-generating pairs depend on n and f and do *not* depend on k, ℓ, R and P . For technical reasons, it is customary to concentrate on *nonnegative* cut-generating functions. This paper will also consider only nonnegative cut-generating pairs, i.e., $\psi \geq 0$ and $\pi \geq 0$.

EXAMPLE 1. Let $n = 1$ and $f \in \mathbb{R} \setminus \mathbb{Z}$. Define

$$\psi(r) = \max \left\{ \frac{r}{1 - [f]}, -\frac{r}{[f]} \right\} \quad \pi(p) = \min \left\{ \frac{[p]}{1 - [f]}, \frac{1 - [p]}{[f]} \right\}, \quad (1.4)$$

where $[x] = x - \lfloor x \rfloor$ denotes the fractional part of any real number x . Then (ψ, π) forms a cut-generating pair for f ; i.e., $\sum_{i=1}^k \psi(r_i) s_i + \sum_{j=1}^\ell \pi(p_j) y_j \geq 1$ is a valid inequality for $X_f(R, P)$. In this case $X_f(R, P)$ and the pair (ψ, π) are determined from a single row of the simplex tableaux of the underlying MILP. (1.4) gives the formula for the well-known Gomory Mixed-Integer (GMI) cut [Gom60].

We call a subset B of \mathbb{R}^n *lattice-free* if B is n -dimensional, closed, convex and the interior of B does not contain integer points. If B is a lattice-free set and $f \in \text{int}(B)$, then B can be defined analytically using the *gauge function* of $B - f$, which is the function

$\phi_{B-f}(r) := \inf \left\{ \rho > 0 : \frac{r}{\rho} \in B - f \right\}$. Intuitively, $\phi_{B-f}(r)$ plays the role of the length of r . Note that ϕ_{B-f} satisfies all the properties of the seminorm with the exception of the symmetry $\phi_{B-f}(r) = \phi_{B-f}(-r)$, which need not be fulfilled. Thus, ϕ_{B-f} induces an oriented “distance” on \mathbb{R}^n , where under orientation we mean that the “distance” from $a \in \mathbb{R}^n$ to $b \in \mathbb{R}^n$ need not be equal to the distance from b to a . By the choice of B , the “distance” from f to every point of \mathbb{Z}^n is at least one. It was observed by Johnson [Joh74] that if (ψ, π) is a cut-generating pair for f , then ψ is the gauge function of $B - f$ for some lattice-free set B . Therefore, one approach to obtaining cut-generating pairs is to start with some lattice-free set B with $f \in \text{int}(B)$, let ψ be the gauge function of $B - f$ and find functions π that can be combined with ψ to form a valid cut-generating pair for f . For example, it is not hard to see that for any lattice-free set B with $f \in \text{int}(B)$, (ϕ_{B-f}, ϕ_{B-f}) is a cut-generating pair. Indeed, notice that ϕ_{B-f} shares the following properties of a distance function: *positive homogeneity*, i.e., $\phi_{B-f}(\lambda r) = \lambda \phi_{B-f}(r)$ for every $r \in \mathbb{R}^n$ and $\lambda \geq 0$, and the *triangle inequality* or *subadditivity*, i.e., $\phi_{B-f}(r_1 + r_2) \leq \phi_{B-f}(r_1) + \phi_{B-f}(r_2)$ for every $r_1, r_2 \in \mathbb{R}^n$. Moreover, since B is lattice-free, $\phi_{B-f}(x - f) \geq 1$ for every $x \in \mathbb{Z}^n$. So for any $(s, y) \in X_f(R, P)$, since $\sum_{i=1}^k r_i s_i + \sum_{j=1}^\ell p_j y_j \in \mathbb{Z}^n - f$, we have $1 \leq \phi_{B-f}(\sum_{i=1}^k r_i s_i + \sum_{j=1}^\ell p_j y_j) \leq \sum_{i=1}^k \phi_{B-f}(r_i s_i) + \sum_{j=1}^\ell \phi_{B-f}(p_j y_j) = \sum_{i=1}^k \phi_{B-f}(r_i) s_i + \sum_{j=1}^\ell \phi_{B-f}(p_j) y_j$.

In general, for a particular lattice-free set B with $f \in \text{int}(B)$ and ψ given by the gauge of $B - f$, there exist multiple functions π that can be appended to make (ψ, π) a cut-generating pair. If (ψ, π) is a cut-generating pair, then π is called a *lifting* of ψ . The set of liftings of ψ is partially ordered: we say that a lifting π' for ψ *dominates* another lifting π'' for ψ if $\pi'(r) \leq \pi''(r)$ for every $r \in \mathbb{R}^n$. A *minimal lifting* for ψ is a lifting which is not dominated by another (distinct) lifting for ψ . A simple application of Zorn’s lemma shows that every lifting π of ψ is dominated by some minimal lifting π' of ψ ; see Theorem 1.1 and its proof in [BHKM13].

Computations with cut-generating functions and unique minimal lifting. The main idea behind cut-generating functions is to keep an arsenal of cut generating pairs (ψ, π) that can be efficiently evaluated so that when we have a concrete MILP to solve, we “plug in” r^1, \dots, r^k into ψ and p^1, \dots, p^ℓ into π and we obtain (1.3) as a cutting plane for solving the MILP. Thus, we want ψ and π to be computable quickly, and thus require computational perspectives on the abstract notions of gauge and minimal liftings. This has been the focus of recent research on the corner polyhedron. We introduce these ideas next for motivating our work in this paper.

Lattice-free sets maximal with respect to inclusion are called *maximal lattice-free*. It is known that every maximal lattice-free set in \mathbb{R}^n is a polyhedron and the recession cone of B is a linear space spanned by rational vectors; see [Lov89], [BCCZ10] and [Ave13]. Then B can be given by finitely many linear inequalities in the form

$$B = \{x \in \mathbb{R}^n : a_i \cdot (x - f) \leq 1 \ \forall i \in I\}, \quad (1.5)$$

where I is a nonempty finite index set with at most 2^n elements (the fact that $|I| \leq 2^n$ is a theorem due to Lovasz [Lov89] - see also [Doi73] and [Sca77]). An important observation in recent work is that, using the fact that the recession cone of B is a linear space, the gauge

function ϕ_{B-f} is computable by means of the simple formula

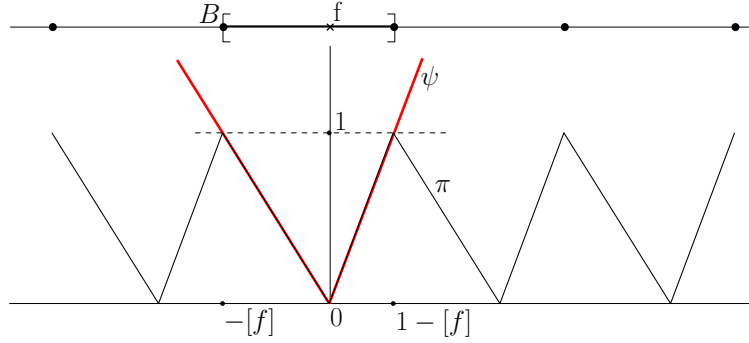
$$\phi_{B-f}(r) = \max \{a_i \cdot r : i \in I\}. \quad (1.6)$$

Given a maximal lattice-free set B , define the following \mathbb{Z}^n -periodic function derived from its gauge function ϕ_{B-f} :

$$\phi_{B-f}^*(r) := \inf_{w \in \mathbb{Z}^n} \phi_{B-f}(r + w). \quad (1.7)$$

It is well-known that if ψ is the gauge ϕ_{B-f} of a lattice-free set B with $f \in \text{int}(B)$ and $\pi = \phi_{B-f}^*$, then (ψ, π) is a cut-generating pair. To see this, observe that $X_f(R, P) = X_f(R, P+W)$ for any integral matrix W . Consequently, the cut-generating pair $(\phi_{B-f}, \phi_{B-f}^*)$ can be strengthened to $(\phi_{B-f}, \phi_{B-f}^*)$.

EXAMPLE 1 (continued). In Example 1, consider the closed interval with endpoints $\lfloor f \rfloor, \lceil f \rceil$ to be the lattice-free set B (recall that we are working in $n = 1$). Then, ψ, π from the example are given by formulas (1.6) and (1.7) respectively, as illustrated in this figure:



Not all cut-generating pairs (ψ, π) are of the form where ψ is a gauge function of a maximal lattice-free convex set, and π is a minimal lifting of ψ - for some cut generating pairs ψ may be the gauge of a lattice-free set that is not maximal. However, we have a nice formula for ψ given by (1.6) when it is the gauge of a maximal lattice-free convex set; thus, one can compute very quickly the coefficients in a valid inequality. This is one reason to study minimal liftings of such special functions ψ . The hope is that minimal liftings may also often have closed form expressions that can be computed efficiently and thus the coefficients in (1.3) can be computed quickly. However, not many efficient procedures for computing minimal liftings are known in the literature. In fact, to the best of the authors' knowledge, (1.7) is the only expression that has been put forward in the literature as a way to compute liftings. The following proposition makes precise the claim that formula (1.7) is efficiently computable when n is not part of the input (meaning, in practice, that n is small). The proof of this proposition appears in the appendix.

PROPOSITION 1.1. Let $n \in \mathbb{N}$, $f \in \mathbb{R}^n \setminus \mathbb{Z}^n$ and B be a maximal lattice-free polyhedron given by (1.5), where I is a nonempty finite index set. Then the infimum in (1.7) is attained for some $w \in \mathbb{Z}^n$. Furthermore, for each fixed dimension $n \in \mathbb{N}$, the computational problem

of determining $\phi_{B-f}^*(r)$ in (1.7) from the input consisting of the rational vectors $a_i \in \mathbb{Q}^n$ with $i \in I$ and the point $f \in \mathbb{Q}^n \setminus \mathbb{Z}^n$, given in the standard binary encoding, is solvable in polynomial time.

Recently, many authors have studied properties of minimal liftings; see [BCC⁺13, BCK12, CCZ11b, DW10b, DW10a]. Given a maximal lattice-free polyhedron B with $f \in \text{int}(B)$, there may exist multiple minimal liftings for the gauge function ϕ_{B-f} , or there may be a unique minimal lifting. We say that a maximal lattice-free set B has the *unique-lifting property* with respect to $f \in \text{int}(B)$ if the gauge function of $B-f$ has exactly one minimal lifting. Otherwise, the gauge function has more than one minimal lifting and we say that B has the *multiple-lifting property* with respect to f . Maximal lattice-free sets with the unique-lifting property give rise to concrete formulas for cutting planes, because ϕ_{B-f}^* defined in (1.7) becomes the unique minimal lifting, and we can use the formula (1.6) for ϕ_{B-f} and Proposition 1.1 to quickly compute the coefficients in (1.3).

Moreover, the function ϕ_{B-f}^* given by (1.7) is not always a minimal lifting of ϕ_{B-f} . The following proposition shows the importance of unique liftings in this context (the proof appears in the appendix).

PROPOSITION 1.2. *Let B be a maximal lattice-free polyhedron with $f \in \text{int}(B)$. Then ϕ_{B-f}^* defined by (1.7) is a minimal lifting for the gauge function ϕ_{B-f} of $B-f$ if and only if ϕ_{B-f} has a unique minimal lifting.*

We hope the above discussion lends credence to the claim that characterizing pairs B, f with unique minimal liftings is an important question in the cut-generating function approach to cutting planes. The purpose of this manuscript is to study maximal lattice-free sets with the unique-lifting property.

Our contributions. We summarize our main contributions in this paper.

- (i) *Invariance result.* A natural question arises: is it possible that B has the unique-lifting property with respect to one $f_1 \in \text{int}(B)$, and has the multiple-lifting property with respect to another $f_2 \in \text{int}(B)$? This question was investigated in [BCK12] and the main result was to establish that this cannot happen when B is a *simplicial* polytope. We prove this for general maximal lattice-free polytopes without the simpliciality assumption: for every maximal lattice-free set B either B has the unique-lifting property with respect to every $f \in \text{int}(B)$ or otherwise B has the multiple-lifting property with respect to every such f (see Theorem 3.6). In view of this result, we can speak about the unique-lifting property of B , without reference to any $f \in \text{int}(B)$.
- (ii) *Result on the volume of the lifting region modulo \mathbb{Z}^n .* To prove the mentioned invariance result, we study the so-called *lifting region* (defined precisely in Section 2), and show that its volume modulo the lattice \mathbb{Z}^n , is an affine function of f (see Theorem 3.4). This is also an extension of the corresponding theorem from [BCK12] for simplicial B . Besides handling the general case, our proof is also significantly shorter and more elegant. We develop a tool for computing volumes modulo \mathbb{Z}^n , which enables us to circumvent a complicated inclusion-exclusion argument from [BCK12] (see pages 349–350 in [BCK12]).

- (iii) *Topological result.* In Section 4 we show that in the space of all maximal lattice-free sets, endowed with the Hausdorff metric, the subset of the sets having the unique-lifting property is closed (see Theorem 4.1). This topological property turns out to be useful for verification of the unique-lifting property of maximal lattice-free sets built using the coproduct operations (see below).
- (iv) *Constructions involving the coproduct operation.* Our techniques give an iterative procedure to construct new families of polytopes with the unique-lifting property in every dimension $n \in \mathbb{N}$. This vastly expands the known list of polytopes with the unique-lifting property. Furthermore, the coproduct operation enables to construct *all* sets with the unique-lifting property for $n = 2$. See Section 5; in particular, Theorem 5.3 and Corollaries 5.4, 5.5 and 5.7.
- (v) *A characterization for special polytopes.* A major contribution of [BCK12] was to characterize the unique-lifting property for a special class of simplices. We generalize all the results from [BCK12] to a broader class of polytopes called *pyramids*, which are constructed using the so-called coproduct operation; see Theorem 6.5 and Theorem 6.3. For these generalizations, we build tools in Section 6 that rely on nontrivial theorems from the geometry of numbers and discrete geometry, such as the Venkov-Alexandrov-McMullen theorem for translative tilings in \mathbb{R}^n and McMullen's characterizations of polytopes with centrally symmetric faces [McM70].

All results that are stated in this paper for general maximal lattice-free polytopes, also hold for maximal lattice-free polyhedra B that are unbounded. This follows from the fact that for unbounded maximal lattice-free sets, the lifting region can be viewed as a cylinder over the lifting region of a lower dimensional maximal lattice-free polytope. Restricting to polytopes keeps the presentation less technical.

In our arguments we rely on tools from convex geometry, the theory of polytopes and the geometry of numbers; for the background information see [Gru07], [Roc70], [Sch93], [Zie95]. The necessary basic notation and terminology, used throughout the manuscript, is introduced in the beginning of the following section.

2 Preliminaries

Basic notation and terminology. Let $n \in \mathbb{N}$. We will also use e_1, e_2, \dots, e_n to denote the standard unit vectors of \mathbb{R}^n . The notation vol stands for the n -dimensional volume (i.e., the Lebesgue measure) in \mathbb{R}^n with the standard normalization $\text{vol}([0, 1]^n) = 1$. For $a, b \in \mathbb{R}^n$ we introduce the segment $[a, b]$ and the relatively open segment (a, b) joining a and b by:

$$\begin{aligned} [a, b] &:= \{(1 - \lambda)a + \lambda b : 0 \leq \lambda \leq 1\}, \\ (a, b) &:= \{(1 - \lambda)a + \lambda b : 0 < \lambda < 1\}. \end{aligned}$$

We will denote the convex hull, affine hull, interior of a set X and the relative interior of a convex set X by $\text{conv}(X)$, $\text{aff}(X)$, $\text{int}(X)$ and $\text{relint}(X)$, respectively.

Given a polytope P , we denote by $\mathcal{F}(P)$ the set of all faces of P . For an integer $i \geq -1$, by $\mathcal{F}^i(P)$ we denote the set of all i -dimensional faces of P . Note that $\mathcal{F}^{-1}(P) = \{\emptyset\}$ and

$\mathcal{F}^n(P) = \{P\}$. Elements of $\mathcal{F}^0(P)$ are called *vertices* of P . Elements of $\mathcal{F}^i(P)$ with $i = \dim(P) - 1$ are called *facets* of P .

We shall make use of the following types of polytopes. Let P be a polytope in \mathbb{R}^n :

- P is called a *pyramid* if P can be represented by $P = \text{conv}(P_0 \cup \{a\})$, where P_0 is a polytope and a is a point lying outside $\text{aff}(P_0)$. In this case the polytope P_0 is called the *base* of P and a is called the *apex* of P_0 .
- P is called a *double pyramid* if P can be represented by $P = \text{conv}(P_0 \cup \{a_1, a_2\})$, where P_0 is a polytope and a_1, a_2 are points such that $[a_1, a_2]$ intersects P_0 , but neither a_1 nor a_2 is in the affine hull of P_0 .
- P is called a *spindle* if P can be represented by $P = (P_1 + a_1) \cap (P_2 + a_2)$, where P_1, P_2 are pointed polyhedral cones and a_1, a_2 are points satisfying $a_i \in \text{relint}(P_j) + a_j$ for $\{i, j\} = \{1, 2\}$. The points a_1, a_2 are called the *apexes* of P .

Geometric characterization of unique minimal liftings. The authors of [BCC⁺13] were able to characterize the unique lifting property in a purely geometric way. Let B be a maximal lattice-free polytope in \mathbb{R}^n and let $f \in B$ (not necessarily in the interior of B). With each $F \in \mathcal{F}(B) \setminus \{\emptyset, B\}$ and f we associate $\text{conv}(\{f\} \cup F)$, which is a pyramid of dimension $\dim(F) + 1$ whenever $f \notin F$. For every $z \in F \cap \mathbb{Z}^n$ we define the polytope

$$S_{F,z}(f) := \text{conv}(\{f\} \cup F) \cap (z + f - \text{conv}(\{f\} \cup F)),$$

given as the intersection of $\text{conv}(\{f\} \cup F)$ and the reflection of $\text{conv}(\{f\} \cup F)$ with respect to $(z + f)/2$. Note that, if $f \notin F$ (which is a generic situation), then $S_{F,z}(f)$ is a spindle. Furthermore, we define

$$R_F(f) := \bigcup_{z \in F \cap \mathbb{Z}^n} S_{F,z}(f),$$

the union of all sets $S_{F,z}(f)$ arising from the face F .

The set

$$R(B, f) := \bigcup_{F \in \mathcal{F}(B) \setminus \{\emptyset, B\}} R_F(f)$$

is called the *lifting region* of B associated with the point f . In [BCC⁺13] it was shown that for $f \in \text{int}(B)$

$$B \text{ has the unique-lifting property with respect to } f \iff R(B, f) + \mathbb{Z}^n = \mathbb{R}^n.$$

Thus, in the rest of the paper, we study the covering properties of $R(B, f)$ by lattice translates to analyze the unique-lifting property of B with respect to $f \in \text{int}(B)$.

We observe that, since $R_{F_1}(f) \subseteq R_{F_2}(f)$ for $F_1, F_2 \in \mathcal{F}(B) \setminus \{B, \emptyset\}$ satisfying $F_1 \subseteq F_2$, the lifting region $R(B, f)$ can also be represented using the set $\mathcal{F}^{n-1}(B)$ of all facets of B as follows:

$$R(B, f) = \bigcup_{F \in \mathcal{F}^{n-1}(B)} R_F(f).$$

3 Invariance theorem on the uniqueness of lifting

Integral formula for the volume of a region modulo \mathbb{Z}^n . For $t \in \mathbb{R}$ let $[t] = t - \lfloor t \rfloor$ be the fractional part of t . For any set $X \subseteq \mathbb{R}^n$, define $X/\mathbb{Z}^n := \{[x] : x \in X\} \subseteq [0, 1]^n$. Observe that a compact set $X \subseteq \mathbb{R}^n$ covers \mathbb{R}^n by lattice translations, i.e., $X + \mathbb{Z}^n = \mathbb{R}^n$ if and only if $\text{vol}(X/\mathbb{Z}^n) = 1$.

Let X be a finite subset of \mathbb{R}^n and assume that we wish to count the number of elements in X/\mathbb{Z}^n . Then it suffices to consider all $x \in X$ and count the element x with the weight $1/|X \cap (x + \mathbb{Z}^n)|$, because all elements of $X \cap (x + \mathbb{Z}^n)$ also belong to X and generate the same element $[x]$. The following lemma is a “continuous counterpart” of the above combinatorial observation.

LEMMA 3.1. *Let $R \subseteq \mathbb{R}^n$ be a compact set with nonempty interior. Then*

$$\text{vol}(R/\mathbb{Z}^n) = \int_R \frac{dx}{|R \cap (x + \mathbb{Z}^n)|}. \quad (3.1)$$

Proof. We use the following formula for substitution of integration variables in the case that the underlying substitution function $f : R \rightarrow \mathbb{R}^n$ is not necessarily injective:

$$\int_{\mathbb{R}^n} \sum_{\tilde{x} \in R : f(\tilde{x})=y} g(\tilde{x}) \, dy = \int_R g(x) |\det(\nabla f)(x)| \, dx. \quad (3.2)$$

Here $g : \mathbb{R}^n \rightarrow \mathbb{R}$ is a Lebesgue measurable function, $f : R \rightarrow \mathbb{R}^n$ is an almost everywhere Lipschitz function and ∇f denotes the Jacobian matrix of f . Note that in the case of injective f , we get a well-known substitution formula with the left hand side equal to $\int_{f(R)} g(f^{-1}(y)) \, dy$. Formula (3.2) is a standard fact in geometric measure theory; it is a special case of Corollary 5.1.3 in [KP08]. We use (3.2) in the case $f(x) = ([x_1], \dots, [x_n])$ and $g(x) = \frac{1}{|R \cap (x + \mathbb{Z}^n)|}$, where $x = (x_1, \dots, x_n) \in R$. Clearly, $(\nabla f)(x)$ is the identity matrix for almost every $x \in R$. Thus, the right hand side of (3.2) coincides with the right hand side of (3.1). We analyze the left hand side of (3.2). Consider an arbitrary $y \in f(R)$, that is, $y \in \mathbb{R}^n$ and $f(x) = y$ for some $x \in R$. We fix x as above. For every $\tilde{x} \in R$ the equality $f(\tilde{x}) = y$ can be written as $f(\tilde{x}) = f(x)$, which is equivalent to $x - \tilde{x} \in \mathbb{Z}^n$. Consequently, $g(x) = g(\tilde{x})$ and $\{\tilde{x} \in R : f(\tilde{x}) = y\} = R \cap (x + \mathbb{Z}^n)$. It follows that the sum on the left hand side of (3.2) is equal to 1 for every $y \in f(R)$, and 0 for every $y \notin f(R)$. Thus, the left hand side of (3.2) is equal to $\int_{f(R)} dy$ and, by this, coincides with the left hand side of (3.1). \square

Structure of congruences modulo \mathbb{Z}^n for points of the lifting region.

LEMMA 3.2. *Let $n \in \mathbb{N}$, let B be a maximal lattice-free polytope in \mathbb{R}^n and let $f \in B$. Let $F_1, F_2 \in \mathcal{F}^{n-1}(B)$ and let $z_i \in \text{relint}(F_i) \cap \mathbb{Z}^n$ for $i \in \{1, 2\}$. Suppose $x_1 \in \text{int}(S_{F_1, z_1}(f))$ and $x_2 \in \text{int}(S_{F_2, z_2}(f))$ be such that $x_1 - x_2 \in \mathbb{Z}^n$. Then $F_1 = F_2$ and, furthermore, the vector $x_1 - x_2$ is the difference of two integral points in the relative interior of F_i ($i \in \{1, 2\}$), i.e.:*

$$x_1 - x_2 \in \text{relint}(F_i) \cap \mathbb{Z}^n - \text{relint}(F_i) \cap \mathbb{Z}^n. \quad (3.3)$$

In particular, the vector $x_1 - x_2$ is parallel to the hyperplane $\text{aff}(F_i)$.

Proof. For $i \in \{1, 2\}$, if f , x_i and z_i do not lie on a common line, we introduce the two-dimensional affine space $A_i := \text{aff}\{f, x_i, z_i\}$. Otherwise choose A_i to be any two-dimensional affine space containing f , x_i and z_i . The set $T_i := \text{conv}(F_i \cup \{f\}) \cap A_i$ is a triangle, whose one vertex is f . We denote the other two vertices by a_i and b_i . Observe that a_i, b_i are on the boundary of facet F_i such that the open interval $(a_i, b_i) \subseteq \text{relint}(F_i)$. Since z_i lies on the line segment connecting a_i, b_i and $z_i \in \text{relint}(F_i)$, there exists $0 < \lambda_i < 1$ such that $z_i = \lambda_i a_i + (1 - \lambda_i) b_i$. Since $x_i \in \text{int}(S_{F_i, z_i}(f))$, there exist $0 < \mu_i, \alpha_i, \beta_i < 1$ such that $x_i = \mu_i f + \alpha_i a_i + \beta_i b_i$ and $\mu_i + \alpha_i + \beta_i = 1$. Also, observe that $x_i \in \text{relint}(T_i \cap (z_i + f - T_i))$. Therefore, $\alpha_i < \lambda_i$ and $\beta_i < 1 - \lambda_i$.

Consider first the case $\mu_1 \geq \mu_2$. In this case, $z_2 + x_1 - x_2$ is an integral point, which can be represented by

$$z_2 + x_1 - x_2 = (\mu_1 - \mu_2)f + (\lambda_2 - \alpha_2)a_2 + (1 - \lambda_2 - \beta_2)b_2 + \alpha_1 a_1 + \beta_1 b_1$$

Observe that $(\mu_1 - \mu_2) + (\lambda_2 - \alpha_2) + (1 - \lambda_2 - \beta_2) + \alpha_1 + \beta_1 = 1$, each of the terms in the sum is nonnegative, and the coefficients $\lambda_2 - \alpha_2, 1 - \lambda_2 - \beta_2, \alpha_1, \beta_1$ are strictly positive. Further, if $F_1 \neq F_2$, then for every facet $F \in \mathcal{F}^{n-1}(B)$, at least one of the points a_1, b_1, a_2, b_2 does not lie on the facet F (because $(a_i, b_i) \subseteq \text{relint}(F_i)$); therefore, the point $z_2 + x_1 - x_2$ does not lie on any facet. This would imply that $z_2 + x_1 - x_2 \in \text{int}(B)$ contradicting the fact that B is lattice-free. Therefore, $F_1 = F_2$. Now for every facet $F \neq F_1$ at least one of a_1, b_1, a_2, b_2 is again not on F , and so the point $z_2 + x_1 - x_2$ does not lie on F . Thus, since $\text{int}(B)$ is lattice-free, $z_2 + x_1 - x_2$ must lie on F_1 ; and further $z_2 + x_1 - x_2 \in \text{relint}(F_1)$. Since $F_1 = F_2$, $z_2 \in \text{relint}(F_1)$ otherwise, $S_{F_2, z_2}(f)$ has empty interior. Thus, we obtain that $x_1 - x_2$ is the difference of two integral points in the relative interior of F_1 .

The case $\mu_1 \leq \mu_2$ is similar with the same analysis performed on $z_1 + x_2 - x_1$. \square

Invariance theorem for unique liftings. We now have all the tools to prove our main invariance result about unique minimal liftings. The main idea is to show that the volume of the lifting region modulo \mathbb{Z}^n is the restriction of an affine function.

We first recall a basic fact about affine transformations.

LEMMA 3.3. *Let H be a hyperplane in \mathbb{R}^n and let f^*, f be two points in $\mathbb{R}^n \setminus H$ that lie in the same open halfspace determined by H . Let T be the affine transformation on \mathbb{R}^n , given by $x \mapsto T(x) := Ax + b$ with $A \in \mathbb{R}^{n \times n}$ and $b \in \mathbb{R}^n$, that acts identically on H and sends f^* to f . For every $x \in \mathbb{R}^n$ let $\delta(x)$ denote the Euclidean distance of x to H . Then $\det(A) = \frac{\delta(f)}{\delta(f^*)}$.*

Proof. By changing coordinates using a rigid motion (i.e., applying an orthogonal transformation and a translation such that distances, angles and volumes are preserved), we can assume that $H = \mathbb{R}^{n-1} \times \{0\}$ and $f^*, f \in \mathbb{R}^{n-1} \times \mathbb{R}_{>0}$. For every $a \in \mathbb{R}^{n-1} \times \mathbb{R}_{>0}$, the value $\delta(a)$ is the last component of a . It is easy to see that the linear transformation T_a that keeps e_1, \dots, e_{n-1} unchanged and sends e_n onto a is given by a matrix whose determinant is $\delta(a)$. Since $T = T_f T_{f^*}^{-1}$, we conclude that $\det(A) = \frac{\delta(f)}{\delta(f^*)}$. \square

THEOREM 3.4. *Let $n \in \mathbb{N}$ and let B be a maximal lattice-free polytope in \mathbb{R}^n . Then the function $f \mapsto \text{vol}(R(B, f)/\mathbb{Z}^n)$, acting from B to \mathbb{R} , is the restriction of an affine function.*

Proof. First, we observe that

$$\text{vol}(R(B, f)/\mathbb{Z}^n) = \sum_{F \in \mathcal{F}^{n-1}(B)} \text{vol}(R_F(B, f)/\mathbb{Z}^n) \quad (3.4)$$

because, by Lemma 3.2, for two distinct facets F_1 and F_2 of B , no point of $\text{int}(R_{F_1}(B, f))$ is congruent to a point of $\text{int}(R_{F_2}(B, f))$ modulo \mathbb{Z}^n . Therefore, it suffices to consider an arbitrary $F \in \mathcal{F}^{n-1}(B)$ and show that the mapping $f \in B \mapsto V(f) := \text{vol}(R_F(B, f)/\mathbb{Z}^n)$ is a restriction of an affine function. We fix a vector $f^* \in \text{int}(B)$. For every $f \in B$, let $\delta(f)$ be the Euclidean distance of f to the hyperplane $\text{aff}(F)$. It suffices to show that for every $f \in B$ one has

$$\frac{V(f)}{V(f^*)} = \frac{\delta(f)}{\delta(f^*)}, \quad (3.5)$$

because (3.5) implies $V(f) = \frac{V(f^*)}{\delta(f^*)} \delta(f)$, where $\frac{V(f^*)}{\delta(f^*)}$ is a constant factor and $f \in B \mapsto \delta(f)$ is a restriction of an affine function. From now on, we fix $f \in B$. If $f \in F$, then (3.5) clearly holds, because both $V(f)$ and $\delta(f)$ vanish. Therefore, we assume $f \in B \setminus F$. By Theorem 3.1, we have the following integral expressions for $V(f^*)$ and $V(f)$:

$$V(f^*) = \int_{R_F(f^*)} \frac{dx^*}{|R_F(f^*) \cap (x^* + \mathbb{Z}^n)|}, \quad V(f) = \int_{R_F(f)} \frac{dx}{|R_F(f) \cap (x + \mathbb{Z}^n)|}. \quad (3.6)$$

Consider the bijective affine transformation T which acts identically on $\text{aff}(F)$ and sends f^* to f . We want to relate both integral expressions in (3.6) by changing the integration variables using the transformation T . We will derive (3.5) by substituting $T(x^*)$ for x in the integral expression for $V(f)$. For this purpose, it is sufficient to verify the following three conditions dealing with the domains of integration, the integrands and the determinant of the Jacobian matrix of T , respectively:

1. T maps $R_F(f^*)$ onto $R_F(f)$, i.e., $T(R_F(f^*)) = R_F(f)$.
2. One has $R_F(f) \cap (x + \mathbb{Z}^n) = R_F(f^*) \cap (x^* + \mathbb{Z}^n)$ if $x = T(x^*)$ and $x^* \in R_F(f^*)$.
3. One has $|\det \nabla T| = \frac{\delta(f)}{\delta(f^*)}$, where ∇T denotes the Jacobian matrix of T .

For verifying Condition 1 we recall that the set $R_F(f)$ is defined using the pyramids $\text{conv}(F \cup \{f\})$ and the reflected pyramids $f + z - \text{conv}(F \cup \{f\})$ with $z \in F \cap \mathbb{Z}^n$. Clearly, T maps $\text{conv}(F \cup \{f^*\})$ onto $\text{conv}(F \cup \{f\})$. But then T also maps $f^* + z - \text{conv}(F \cup \{f^*\})$ with $z \in F \cap \mathbb{Z}^n$ onto $f + z - \text{conv}(F \cup \{f\})$ because an element of $f^* + z - \text{conv}(F \cup \{f^*\})$ is an affine combination of f^* , z and a point of $\text{conv}(F \cup \{f^*\})$ with coefficients 1, 1 and -1 . Furthermore, from the definition of T we get $T(f^*) = f$, $T(z) = z$ and $T(\text{conv}(F \cup \{f^*\})) = \text{conv}(F \cup \{f\})$. Thus, $T(R_F(f^*)) = R_F(f)$ and Condition 1 is fulfilled.

We verify Condition 2. Choose an arbitrary $y^* \in R_F(f^*) \cap (x^* + \mathbb{Z}^n)$. Then $x^*, y^* \in R_F(f^*)$, $x^* - y^* \in \mathbb{Z}^n$ and, by Lemma 3.2, the segment $[x^*, y^*]$ (which is possibly degenerate to a point) is parallel to $\text{aff}(F)$. Since T acts identically on $\text{aff}(F)$ and $[x^*, y^*]$ is parallel to $\text{aff}(F)$, the segment $[T(x^*), T(y^*)]$ is a translation of $[x^*, y^*]$. In particular, $T(y^*) - T(x^*) \in \mathbb{Z}^n$ and by this, taking into account Condition 1, we see that $T(y^*)$ is an element of $T(R_F(f^*)) \cap (T(x^*) + \mathbb{Z}^n) = R_F(f) \cap (x + \mathbb{Z}^n)$. This shows that the image of $R_F(f^*) \cap (x^* + \mathbb{Z}^n)$ under T

is a subset of $R_F(f) \cap (x + \mathbb{Z}^n)$. Interchanging x and x^* and replacing T by T^{-1} , the above argument can also be used to show that the image of $R_F(f) \cap (x + \mathbb{Z}^n)$ under T^{-1} is a subset of $R_F(f^*) \cap (x^* + \mathbb{Z}^n)$. The latter verifies Condition 2.

Condition 3 is fulfilled in view of Lemma 3.3.

The above shows (3.5) and yields the conclusion. \square

Theorem 3.4 implies the following.

COROLLARY 3.5. *Let B be a maximal lattice-free polytope in \mathbb{R}^n . Then the set $\{f \in B : \text{vol}(R(B, f)/\mathbb{Z}^n) = 1\}$ is a face of B .*

Proof. Since $\text{vol}(R(B, f)/\mathbb{Z}^n)$ is always at most 1, the value 1 is a maximum value for the function $\text{vol}(R(B, f)/\mathbb{Z}^n)$. By Theorem 3.4, optimizing this function over B is a linear program and hence the optimal set is a face of B . \square

THEOREM 3.6. (Unique-lifting invariance theorem.) *Let $n \in \mathbb{N}$ and let B be a maximal lattice-free polytope in \mathbb{R}^n . Let $f_1, f_2 \in \text{int}(B)$. Then B has the unique-lifting property with respect to f_1 if and only if B has the unique-lifting property with respect to f_2 .*

Proof. Corollary 3.5 implies Theorem 3.6. Indeed, if the set $\{f \in B : \text{vol}(R(B, f)/\mathbb{Z}^n) = 1\}$ is B , then $R(B, f) + \mathbb{Z}^n = \mathbb{R}^n$ for all $f \in B$ and, in particular, for all $f \in \text{int}(B)$. Hence, B has the unique-lifting property with respect to every $f \in \text{int}(B)$. Otherwise, $R(B, f) + \mathbb{Z}^n \neq \mathbb{R}^n$ for all $f \in \text{int}(B)$, which implies that B has the multiple-lifting property with respect to every $f \in \text{int}(B)$. \square

4 Limits of polytopes with the unique-lifting property

Background information on the Hausdorff metric and convex bodies. We first collect standard notions and basic facts from convex geometry that we need for our topological result on unique liftings. We refer the reader to the monograph [Sch93, Chapter 1].

Let \mathcal{C}^n be the family of all nonempty compact subsets of \mathbb{R}^n and \mathcal{K}^n be the family of all nonempty compact *convex* subsets of \mathbb{R}^n . With each $K \in \mathcal{C}^n$ we associate the support function $h(K, \cdot)$ defined by $h(K, u) := \max\{u \cdot x : x \in K\}$. The function $h(K, \cdot)$ is sublinear and, by this, also continuous. Furthermore, $h(K, u)$ is additive in K with respect to the Minkowski addition. That is, if $K, L \in \mathcal{C}^n$ and $u \in \mathbb{R}^n$, then $h(K + L, u) = h(K, u) + h(L, u)$.

As a direct consequence of separation theorems the following characterization of the inclusion-relation for elements in \mathcal{K}^n can be derived: for $K, L \in \mathcal{K}^n$, one has $K \subseteq L$ if and only if the inequality $h(K, u) \leq h(L, u)$ holds for every $u \in \mathbb{R}^n$. The latter, in combination with the additivity of $h(K, u)$ in K , implies the cancellation law for Minkowski addition: if $K, L, M \in \mathcal{K}^n$, then the inclusion $K \subseteq L$ is equivalent to the inclusion $K + M \subseteq L + M$.

For $K \in \mathcal{K}^n$, a set of the form

$$F(K, u) := \{x \in K : u \cdot x = h(K, u)\}, \quad (4.1)$$

where $u \in \mathbb{R}^n$, is called an exposed face of K in direction u . All (nonempty) faces of a polytope are exposed.

Given $c \in \mathbb{R}^n$ and $\rho \geq 0$, let $\mathbb{B}(c, \rho)$ denote the closed (Euclidean) ball of radius ρ with center at c . For $K, L \in \mathcal{C}^n$, the Hausdorff distance $\text{dist}(K, L)$ between K and L is defined by

$$\text{dist}(K, L) := \min \{ \rho \geq 0 : K \subseteq L + \mathbb{B}(o, \rho), L \subseteq K + \mathbb{B}(o, \rho) \}.$$

The Hausdorff distance is a metric on \mathcal{C}^n . The formulation of the main theorem of this section involves the topology induced by the Hausdorff distance. In what follows, speaking about the convergence for sequences of elements from \mathcal{C}^n , we shall always mean the convergence in the Hausdorff metric.

It is known that the mapping $h(K, u)$ continuously depends on the pair $(K, u) \in \mathcal{K}^n \times \mathbb{R}^n$; see the comment preceding Lemma 1.8.10 in [Sch93]. More precisely, if $(K_t)_{t \in \mathbb{N}}$ is a sequence of elements of \mathcal{K}^n converging to some $K \in \mathcal{K}^n$ and $(u_t)_{t \in \mathbb{N}}$ is a sequence of vectors in \mathbb{R}^n converging to some vector $u \in \mathbb{R}^n$, then $h(K_t, u_t)$ converges to $h(K, u)$, as $t \rightarrow \infty$. Furthermore, one can pass to the limit in inclusions, with respect to convergence in the Hausdorff distance. More precisely, let $(K_t)_{t \in \mathbb{N}}$ and $(L_t)_{t \in \mathbb{N}}$ be sequences of elements from \mathcal{C}^n converging to $K \in \mathcal{C}^n$ and $L \in \mathcal{C}^n$. If $K_t \subseteq L_t$ for every $t \in \mathbb{N}$, then $K \subseteq L$. In particular, considering the case that K_t consists of a single point, say p_t , we derive that if $p_t \in L_t$ for every $t \in \mathbb{N}$ and, p_t converges to some $p \in \mathbb{R}^n$, as $t \rightarrow \infty$, then we have $p \in L$.

We also note that, if $K, L \in \mathcal{C}^n$, the definition of the Hausdorff distance implies

$$\text{dist}(\text{conv}(K), \text{conv}(L)) \leq \text{dist}(K, L). \quad (4.2)$$

Topology of the space of polytopes with the unique-lifting property. The following is the main result of this section.

THEOREM 4.1. *Let $(B_t)_{t \in \mathbb{N}}$ be a convergent sequence (in the Hausdorff metric) of maximal lattice-free polytopes in \mathbb{R}^n such that the limit B of this sequence is a maximal lattice-free polytope. If, for every $t \in \mathbb{N}$, the set B_t has the unique-lifting property, then B too has the unique-lifting property.*

Proof. Fix an arbitrary $f \in \text{int}(B)$. Let $R := R(B, f)$. We need to verify $R + \mathbb{Z}^n = \mathbb{R}^n$.

Choose $\varepsilon > 0$ such that $\mathbb{B}(f, \varepsilon) \subseteq B$. Let us show that $f \in B_t$ for all sufficiently large $t \in \mathbb{N}$ and, thus, the lifting region $R_t := R(B_t, f)$ is well-defined for all sufficiently large t . Since B is the limit of B_t , as $t \rightarrow \infty$, there exists $t_0 \in \mathbb{N}$ such that $B \subseteq B_t + \mathbb{B}(o, \varepsilon)$ for all $t \geq t_0$. Hence, $f + \mathbb{B}(o, \varepsilon) = \mathbb{B}(f, \varepsilon) \subseteq B \subseteq B_t + \mathbb{B}(o, \varepsilon)$ for all $t \geq t_0$. Using the cancellation law for the Minkowski addition, we arrive at $f \in B_t$ for all $t \geq t_0$. Thus, replacing $(B_t)_{t \in \mathbb{N}}$ by its appropriate subsequence, we assume that $f \in B_t$ for every $t \in \mathbb{N}$.

Assume that, for every $t \in \mathbb{N}$, the set B_t has the unique-lifting property. By Theorem 3.4, one has $R_t + \mathbb{Z}^n = \mathbb{R}^n$. Below, we use the latter relation to show $R + \mathbb{Z}^n = \mathbb{R}^n$. We consider an arbitrary $x \in \mathbb{R}^n$ and show that $x \in R + \mathbb{Z}^n$.

Since $x \in \mathbb{R}^n = R_t + \mathbb{Z}^n$, there exists $w_t \in \mathbb{Z}^n$ such that $x \in R_t + w_t$ for every $t \in \mathbb{N}$. Every convergent sequence of nonempty compact sets is necessarily bounded. Hence, there exists $M \in \mathbb{N}$ such that every B_t with $t \in \mathbb{N}$ is a subset of the box $[-M, M]^n$. In view of the inclusions $R_t \subseteq B_t \subseteq [-M, M]^n$, the vector w_t lies in the finite set $([-M, M]^n + x) \cap \mathbb{Z}^n$. Consequently, replacing $(B_t)_{t \in \mathbb{N}}$ by its appropriate subsequence, we can assume that w_t is independent of t . With this assumption, we have $x \in R_t + w$ for every $t \in \mathbb{N}$, where $w := w_t$.

The point $x - w$ belongs to R_t . Using the definition of the lifting region, we conclude that $x - w \in \text{conv}(\{f\} \cup F_t) \cup (f + z_t - \text{conv}(\{f\} \cup F_t))$ for some face F_t of B_t and some point $z_t \in F_t \cap \mathbb{Z}^n$. Note that z_t lies in the finite set $[-M, M]^n \cap \mathbb{Z}^n$. Passing to appropriate subsequences once again, we can assume that z_t is independent of t . With this assumption, for every $t \in \mathbb{N}$, we have

$$x - w \in \text{conv}(\{f\} \cup F_t) \cap (f + z - \text{conv}(\{f\} \cup F_t)), \quad (4.3)$$

where $z := z_t$. Relation (4.3) implies that both $x - w$ and $(f + z) - (x - w)$ belong to $\text{conv}(\{f\} \cup F_t)$. The latter can be represented by the equalities

$$x - w = (1 - \lambda_t)f + \lambda_t p_t, \quad (4.4)$$

$$(f + z) - (x - w) = (1 - \mu_t)f + \mu_t q_t, \quad (4.5)$$

which hold for some $\lambda_t, \mu_t \in [0, 1]$ and some $p_t, q_t \in F_t$. We can represent F_t as $F_t = F(B_t, u_t)$ for some unit vector u_t in \mathbb{R}^n . The conditions $p_t \in F_t$ and $q_t \in F_t$ can be reformulated as follows:

$$p_t \in B_t, \quad h(B_t, u_t) = u_t \cdot p_t, \quad (4.6)$$

$$q_t \in B_t, \quad h(B_t, u_t) = u_t \cdot q_t. \quad (4.7)$$

Since λ_t, μ_t lie in the compact set $[0, 1]$, the points p_t, q_t lie in the compact set $[-M, M]^n$ and u_t lie in the unit sphere, we can pass to a subsequence and assume that the scalars $\lambda_t, \mu_t \in [0, 1]$, the points p_t, q_t and the unit vector u_t converge to some scalars $\lambda, \mu \in [0, 1]$, points $p \in \mathbb{R}^n$, $q \in \mathbb{R}^n$ and a unit vector u , respectively, as $t \rightarrow \infty$. Passing to the limit, as $t \rightarrow \infty$, in relations (4.4)–(4.7), we arrive at the respective relations

$$\begin{aligned} x - w &= (1 - \lambda)f + \lambda p, \\ (f + z) - (x - w) &= (1 - \mu)f + \mu q \end{aligned}$$

and

$$\begin{aligned} p &\in B, & h(B, u) &= u \cdot p, \\ q &\in B, & h(B, u) &= u \cdot q. \end{aligned}$$

The latter implies that $x - w \in \text{conv}(\{f\} \cup F) \cap (f + z - \text{conv}(\{f\} \cup F))$ for $F := F(B, u)$. Thus, $x - w$ belongs to R and, by this, x belongs to $R + \mathbb{Z}^n$. Since x was chosen arbitrarily, we arrive at $\mathbb{R}^n = R + \mathbb{Z}^n$. It follows that B has the unique-lifting property. \square

Let \mathcal{U}^n be the set of all maximal lattice-free polytopes in \mathbb{R}^n with the unique-lifting property and let \mathcal{M}^n be the set of all maximal lattice-free polytopes in \mathbb{R}^n . We view \mathcal{U}^n and \mathcal{M}^n as metric spaces endowed with the Hausdorff metric. Clearly, $\mathcal{U}^n \subseteq \mathcal{M}^n$. Theorem 4.1 asserts that \mathcal{U}^n is a *closed* subset of \mathcal{M}^n .

In view of Theorem 4.1, one may wonder if the limit of every convergent sequence of maximal lattice-free polytopes with the unique-lifting property is necessarily a maximal lattice-free set. This question is beyond the scope of this manuscript; however, we note that the limit of a convergent sequence of *arbitrary* maximal lattice-free polytopes is not necessarily a maximal

lattice-free polytope. In other words, \mathcal{M}^n is not a closed subset of \mathcal{K}^n . For every $n \geq 3$, this can be shown by considering a sequence of maximal lattice-free sets which get arbitrarily flat and in the limit become non-full-dimensional, following a construction from [Ave12, Example 3.4] (we recall that according to our definition, maximal lattice-free sets in \mathbb{R}^n are required to be n -dimensional). In the case $n = 2$ one can show that the square $[0, 1]^2$, which is lattice-free but not maximal lattice-free, lies in the closure of \mathcal{M}^2 . Let c be the center of $[0, 1]^2$. Rotating the square by a small angle around c and slightly enlarging the rotated square by a homothetic transformation with homothetic center at c , we can construct a maximal lattice-free polytope from \mathcal{M}^2 which is arbitrarily close to $[0, 1]^2$. This shows that $[0, 1]^2$ belongs to the closure of \mathcal{M}^2 . Hence, \mathcal{M}^2 is not a closed subset of \mathcal{K}^2 .

5 Construction of polytopes with the unique-lifting property

Coproduct and its properties. Recall that $\mathcal{F}(P)$ denotes the set of all faces of P , $\mathcal{F}^i(P)$ denotes the set of faces of dimension i and $\mathcal{F}^{\dim(P)-1}(P)$ denotes the set of facets of P . Further, $h(P, u) := \max \{u \cdot x : x \in P\}$ denotes the support function of the polytope P , and we will use the notation $F(P, u) = \{x \in P : h(P, u) = u \cdot x\}$ to be the optimal face of P when maximizing in the direction of u .

Let $n_1, n_2 \in \mathbb{N}$ and $n := n_1 + n_2$. For each $i \in \{1, 2\}$, let o_i be the origin of \mathbb{R}^{n_i} .

Given $K_1 \in \mathcal{K}^{n_1}$ and $K_2 \in \mathcal{K}^{n_2}$, the set

$$K_1 \diamond K_2 := \text{conv}(K_1 \times \{o_2\} \cup \{o_1\} \times K_2) \in \mathcal{K}^n.$$

is called the *coproduct* of K_1 and K_2 ¹. Clearly, up to nonsingular affine transformations, pyramids and double pyramids can be given as coproducts $K_1 \diamond K_2$ for polytopes K_1, K_2 with $o_i \in K_i$ for each $i \in \{1, 2\}$ and $\dim(K_1) = 1$. Note also that the coproduct operation is associative, and so in the expressions involving the coproduct of three and more sets we can omit brackets. We shall use coproducts of an arbitrary number of sets later in this section.

Clearly,

$$K_1 \diamond K_2 = \bigcup_{0 \leq \lambda \leq 1} (1 - \lambda)K_1 \times \lambda K_2. \quad (5.1)$$

By the basic properties of the relative-interior operation (see [Roc70, Theorem 6.9]), we have

$$\text{relint}(K_1 \diamond K_2) = \bigcup_{0 < \lambda < 1} (1 - \lambda) \text{relint}(K_1) \times \lambda \text{relint}(K_2). \quad (5.2)$$

If $\dim(K_i) = n_i$ for each $i \in \{1, 2\}$, then $\dim(K_1 \diamond K_2) = n$ and the operation relint in (5.2) can be replaced by int .

LEMMA 5.1. (On faces of the coproduct of polytopes.) *For $i \in \{1, 2\}$, let $n_i \in \mathbb{N}$, let o_i be the origin of \mathbb{R}^{n_i} and let P_i be an n_i -dimensional polytope in \mathbb{R}^{n_i} with $o_i \in P_i$. Let $P := P_1 \diamond P_2 \subseteq \mathbb{R}^n$, where $n := n_1 + n_2$. Let F be a nonempty subset of \mathbb{R}^n . Then the following assertions hold:*

¹[HRGZ97, p. 250] calls this construction the *free sum*; we use *coproduct* following a suggestion by Peter McMullen. The construction is dual to the operation of taking Cartesian products, i.e., when $o_i \in \text{int}(K_i)$ for each $i \in \{1, 2\}$, we have the relation $(K_1 \times K_2)^\circ = K_1^\circ \diamond K_2^\circ$ for the polar polytopes of $K_1 \times K_2$, K_1 and K_2 .

(a) The set F is a face of P if and only if one of the following four conditions is fulfilled:

$$F = F_1 \diamond F_2, \quad \text{where} \quad o_i \notin F_i \in \mathcal{F}(P_i) \quad \forall i \in \{1, 2\}, \quad \text{or} \quad (5.3)$$

$$F = F_1 \diamond F_2, \quad \text{where} \quad o_i \in F_i \in \mathcal{F}(P_i) \quad \forall i \in \{1, 2\}, \quad \text{or} \quad (5.4)$$

$$F = F_1 \times \{o_2\}, \quad \text{where} \quad o_1 \notin F_1 \in \mathcal{F}(P_1), \quad \text{or} \quad (5.5)$$

$$F = \{o_1\} \times F_2, \quad \text{where} \quad o_2 \notin F_2 \in \mathcal{F}(P_2). \quad (5.6)$$

(b) Under conditions (5.3), (5.4), (5.5) and (5.6), the dimension of the face F of P is expressed by the equalities $\dim(F) = \dim(F_1) + \dim(F_2) + 1$, $\dim(F) = \dim(F_1) + \dim(F_2)$, $\dim(F) = \dim(F_1)$ and $\dim(F) = \dim(F_2)$, respectively.

(c) The set F is a facet of P if and only if one of the following three conditions is fulfilled:

$$F = F_1 \diamond F_2, \quad \text{where} \quad o_i \notin F_i \in \mathcal{F}^{n_i-1}(P_i) \quad \forall i \in \{1, 2\}, \quad \text{or} \quad (5.7)$$

$$F = F_1 \diamond P_2, \quad \text{where} \quad o_1 \in F_1 \in \mathcal{F}^{n_1-1}(P_1), \quad \text{or} \quad (5.8)$$

$$F = P_1 \diamond F_2, \quad \text{where} \quad o_2 \in F_2 \in \mathcal{F}^{n_2-1}(P_2). \quad (5.9)$$

Proof. Assertion (a): We start with the necessity in (a). We assume that F is a face of P and show that one of the four conditions (5.3)–(5.6) is fulfilled. By the assumptions of the lemma, F is nonempty, and so we have

$$F = F(P, u) = \{x \in P : h(P, u) = u \cdot x\}$$

for some $u := (u_1, u_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$. Since

$$h(P, u) = \max\{h(P_1, u_1), h(P_2, u_2)\},$$

we get:

$$F(P, u) = F(P_1, u_1) \diamond F(P_2, u_2) \quad \text{if} \quad h(P_1, u_1) = h(P_2, u_2), \quad (5.10)$$

$$F(P, u) = F(P_1, u_1) \times \{o_2\} \quad \text{if} \quad h(P_1, u_1) > h(P_2, u_2), \quad (5.11)$$

$$F(P, u) = \{o_1\} \times F(P_2, u_2) \quad \text{if} \quad h(P_1, u_1) < h(P_2, u_2). \quad (5.12)$$

Since for each $i \in \{1, 2\}$, one has $o_i \in P_i$, the support function $h(P_i, \cdot)$ is nonnegative. We compare $h(P_1, u_1)$ with $h(P_2, u_2)$ and distinguish three cases as in (5.10)–(5.12).

In the case $h(P_1, u_1) = h(P_2, u_2)$, we define $F_i := F(P_i, u_i)$ for $i \in \{1, 2\}$. One has either $h(P_1, u_1) = h(P_2, u_2) > 0$ or $h(P_1, u_1) = h(P_2, u_2) = 0$. In the former case, the face $F(P_i, u)$ of P_i does not contain o_i for each $i \in \{1, 2\}$ and, by this, (5.3) is fulfilled. In the latter case, the face $F(P_i, u_i)$ of P_i contains o_i for each $i \in \{1, 2\}$ and, by this, (5.4) is fulfilled. If $h(P_1, u_1) > h(P_2, u_2)$, we have $h(P_1, u_1) > 0$ and so (5.5) is fulfilled for $F_1 := F(P_1, u_1)$. Analogously, if $h(P_2, u_1) < h(P_2, u_2)$, we have $h(P_2, u_2) > 0$ and so (5.6) is fulfilled for $F_2 := F(P_2, u_2)$. This proves the necessity in (a).

For proving the sufficiency, we assume that F fulfills one of the four conditions (5.3)–(5.6) and show that $F = F(P, u)$ for an appropriate choice of $u \in \mathbb{R}^n$. Consider the case that F fulfills (5.3). For each $i \in \{1, 2\}$, we choose a vector $u_i \in \mathbb{R}^{n_i}$, with $F_i = F(P_i, u_i)$. Since

$o_i \notin F_i$, we have $h(P_i, u_i) > 0$ and by this also $u_i \neq o_i$. Appropriately rescaling the vectors u_1 and u_2 , we ensure the inequality $h(P_1, u_1) = h(P_2, u_2)$. In view of (5.10), it follows that $F = F(P, u)$ with $u = (u_1, u_2)$. The case that F fulfills (5.4) is similar. For each $i \in \{1, 2\}$ we choose $u_i \in \mathbb{R}^{n_i}$ with $F_i = F(P_i, u_i)$. Since $o_i \in F_i$, we have $h(P_i, u_i) = 0$. In view of (5.10), $F = F(P, u)$ with $u = (u_1, u_2)$. In the case that F fulfills (5.5), we choose $u_1 \in \mathbb{R}^{n_1}$ satisfying $F_1 = F(P_1, u_1)$ and define $u_2 := o_2$. Since $o_1 \notin F_1$, we get $h(P_1, u_1) > 0$. In view of (5.11), we get $F = F(P, u)$ for $u = (u_1, u_2)$. The case that F fulfills (5.6) is completely analogous to the previously considered case.

Assertion (b): The cases (5.5) and (5.6) are trivial. In the case (5.4), the face F contains o , and so the dimension of F is the dimension of the linear hull of F . Applying the definition of the coproduct and the fact that, for each $i \in \{1, 2\}$, the face F_i contains o_i , we see that the linear hull of F is the Cartesian product of the linear hulls of F_1 and F_2 . Hence $\dim(F) = \dim(F_1) + \dim(F_2)$. The case (5.3) can be handled by a reduction to the case (5.4). Assume that (5.3) is fulfilled. Then, in view of (5.1), the set $F = F_1 \diamond F_2$ does not contain o . Then $\dim(K) = \dim(F) + 1$ for $K := \text{conv}(F \cup \{o\})$. Using the definition of the coproduct, we can easily verify the equality $K = \text{conv}(F_1 \cup \{o_1\}) \diamond \text{conv}(F_2 \cup \{o_2\})$. Using the argument from the case (5.4) we conclude that $\dim(K) = \dim(\text{conv}(F_1 \cup \{o_1\})) + \dim(\text{conv}(F_2 \cup \{o_2\}))$. Since $o_i \notin F_i$, we have $\dim(\text{conv}(F_i \cup \{o_i\})) = \dim(F_i) + 1$. Hence $\dim(F) = \dim(F_1) + \dim(F_2) + 1$.

Assertion (c) is a straightforward consequence of (a) and (b). \square

Constructions based on the coproduct operation. It will be more convenient to work with general affine lattices and then specialize to the \mathbb{Z}^n case. To that end, we say that a set $\Lambda \subseteq \mathbb{R}^n$ is an affine lattice of rank n if Λ is a translation of a lattice of rank n . Equivalently, a set $\Lambda \subset \mathbb{R}^n$ is an affine lattice of rank n if and only if there exist affinely independent points $x_0, \dots, x_n \in \mathbb{R}^n$ such that Λ is the set of all $x = z_0 x_0 + \dots + z_n x_n$ with $z_0, \dots, z_n \in \mathbb{Z}$ and $z_0 + \dots + z_n = 1$. Note that the notions of lattice-free set, maximal lattice-free set, lifting region (denoted by $R(B, f)$) and sets with the unique-lifting property, which were introduced with respect the integer lattice \mathbb{Z}^n , can be extended directly to the more general situation where, in place of \mathbb{Z}^n , we take an arbitrary affine lattice Λ of rank n in \mathbb{R}^n . Thus, for such Λ we can introduce the respective notions of Λ -free set, maximal Λ -free set, lifting region with respect to Λ (which we will denote by $R_\Lambda(B, f)$) and set with the unique-lifting property with respect to Λ . We shall use the following result of Lovász [Lov89]; see also [BCCZ10] and [Ave13].

THEOREM 5.2. (Lovász's characterization of maximal lattice-free sets.) *Let B be an n -dimensional Λ -free polyhedron in \mathbb{R}^n . Then B is maximal Λ -free if and only if $\text{relint}(F) \cap \Lambda \neq \emptyset$ for every facet F of B .*

The following is the main result of this section.

THEOREM 5.3. (Coproduct construction of various types of Λ -free sets.) *For $i \in \{1, 2\}$, let $n_i \in \mathbb{N}$, let o_i be the origin of \mathbb{R}^{n_i} , let Λ_i be an affine lattice of rank n_i in \mathbb{R}^{n_i} and let B_i be an n_i -dimensional polytope with $o_i \in B_i$. Let $0 < \mu < 1$. Then, for the n -dimensional polytope $B := B_1 \diamond B_2$ with $n := n_1 + n_2$ and the affine lattice $\Lambda := (1 - \mu)\Lambda_1 \times \mu\Lambda_2$ of rank n , the following assertions hold:*

(a) *If B_i is Λ_i -free for each $i \in \{1, 2\}$, then B is Λ -free.*

- (b) If B_i is maximal Λ_i -free for each $i \in \{1, 2\}$, then B is maximal Λ -free.
- (c) If B_i is maximal Λ_i -free and has the unique-lifting property with respect to Λ_i for each $i \in \{1, 2\}$, then B is maximal Λ -free and has the unique-lifting property with respect to Λ .

Proof. (a): Consider an arbitrary point $x = (x_1, x_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ belonging to $\text{int}(B_1 \diamond B_2)$. In view of (5.2), one has $x_1 \in (1 - \lambda)\text{int}(B_1)$ and $x_2 \in \lambda\text{int}(B_2)$ for some $0 < \lambda < 1$. If $\lambda \geq \mu$, then taking into account $o_1 \in B_1$, we obtain $x_1 \in (1 - \lambda)\text{int}(B_1) \subseteq (1 - \mu)\text{int}(B_1)$. Since B_1 is Λ_1 -free, we have $x_1 \notin (1 - \mu)\Lambda_1$ implying that $x \notin \Lambda$. Analogously, in the case $\lambda \leq \mu$, we deduce that the point x_2 is not in $\mu\Lambda_2$ and thus $x \notin \Lambda$.

(b): Assume that B_i is maximal Λ_i -free for each $i \in \{1, 2\}$. We show that B is maximal Λ -free. In view of (a), the polytope B is Λ -free. By Theorem 5.2, in order to verify the maximality of B it suffices to show that the relative interior of each facet of B contains a point of Λ . Let F be an arbitrary facet of B . We use the classification of facets of the coproduct which is provided by Lemma 5.1.(c). Consider the case that F fulfills condition (5.7). By Theorem 5.2 applied to the maximal Λ_i -free set B_i , for each $i \in \{1, 2\}$ there exists a point $x_i \in \text{relint}(F_i)$ belonging to Λ_i . Then the point $x := ((1 - \mu)x_1, \mu x_2)$ belongs to Λ . By (5.2), the point x also belongs to $\text{relint}(F)$. Let us switch to the case that F fulfills condition (5.8). By Theorem 5.2, there exists a point $x_1 \in \text{relint}(F_1)$ which belongs to Λ_1 . Then $(1 - \mu)x_1 \in (1 - \mu)\text{relint}(F_1)$. Since $o_1 \in F_1$, the latter containment relation remains valid if we slightly shrink the right hand side $(1 - \mu)\text{relint}(F_1)$. That is, $(1 - \mu)x_1 \in (1 - \lambda)\text{relint}(F_1)$ for some λ satisfying $\mu < \lambda < 1$, which is sufficiently close to μ . Since $o_2 \in B_2$, we have $\text{int}(B_2) \subsetneq \frac{\lambda}{\mu}\text{int}(B_2)$. Since B_2 is maximal Λ_2 -free, there exists a point $x_2 \in \frac{\lambda}{\mu}\text{int}(B_2)$ which belongs to Λ_2 . It follows that $\mu x_2 \in \lambda\text{int}(B_2)$ is a point belonging to $\mu\Lambda_2$. Thus, $x := ((1 - \mu)x_1, \mu x_2)$ is a point belonging to $(1 - \lambda)\text{relint}(F_1) \times \lambda\text{int}(B_2)$ and to Λ . Taking into account (5.2), we see that x belongs to $\text{relint}(F)$. The case of F fulfilling condition (5.8) is completely analogous to the previously considered case. Summarizing, we conclude that the relative interior of each facet of B contains a point of Λ . Thus, by Theorem 5.2, the set B is maximal Λ -free.

(c): We distinguish two cases.

Case 1: $o_i \in \text{int}(B_i)$ for each $i \in \{1, 2\}$. In this case $o \in \text{int}(B)$. Thus for showing that B has the unique-lifting property with respect to Λ , it suffices to check the equality $R + \Lambda = \mathbb{R}^n$ for the lifting region

$$R := R_\Lambda(B, o) = \bigcup_{\substack{F \in \mathcal{F}^{n-1}(B) \\ z \in \Lambda \cap F}} S_{F,z}(o), \quad (5.13)$$

where, for every $F \in \mathcal{F}^{n-1}(B)$ and $z \in \Lambda \cap F$, one has

$$S_{F,z}(o) = \text{conv}(F \cup \{o\}) \cap (z - \text{conv}(F \cup \{o\})). \quad (5.14)$$

Since $o_i \in \text{int}(B_i)$, no facet of B_i contains o_i . Thus, by Lemma 5.1.(c), a subset F of \mathbb{R}^n is a facet of B if and only if $F = F_1 \diamond F_2$, where F_i is a facet of B_i for each $i \in \{1, 2\}$. We consider an arbitrary such facet $F = F_1 \diamond F_2$. Choose also an arbitrary $z_i \in F_i \cap \Lambda_i$ for each $i \in \{1, 2\}$ and introduce the point $z := ((1 - \mu)z_1, \mu z_2)$, which by construction belongs to $F \cap \Lambda$. We

establish an inclusion relation between $S_{F,z}(o)$ and the two sets $S_{F,z_i}(o_i)$ with $i \in \{1, 2\}$. The set $\text{conv}(F \cup \{o\})$, which occurs twice on the right hand side of (5.14), fulfills

$$\begin{aligned} \text{conv}(F \cup \{o\}) &= \text{conv}(F_1 \times \{o_2\} \cup \{o_1\} \times F_2 \cup \{o\}) \\ &= \text{conv}((F_1 \cup \{o_1\}) \times \{o_2\} \cup \{o_1\} \times (F_2 \cup \{o_2\})) \\ &= \text{conv}(F_1 \cup \{o_1\}) \diamond \text{conv}(F_2 \cup \{o_2\}) \\ &\supseteq (1 - \mu) \text{conv}(F_1 \cup \{o_1\}) \times \mu \text{conv}(F_2 \cup \{o_2\}). \end{aligned} \quad (5.15)$$

Therefore, also $z - \text{conv}(F \cup \{o\}) \supseteq (1 - \mu)(z_1 - \text{conv}(F_1 \cup \{o_1\})) \times \mu(z_2 - \text{conv}(F_2 \cup \{o_2\}))$. Analogously to (5.14), one has $S_{F_i,z_i}(o_i) = \text{conv}(F_i \cup \{o_i\}) \cap (z_i - \text{conv}(F_i \cup \{o_i\}))$ for each $i \in \{1, 2\}$. From this and (5.15), we get

$$S_{F,z}(o) \supseteq (1 - \mu)S_{F_1,z_1}(o_1) \times \mu S_{F_2,z_2}(o_2).$$

In view of (5.13), the latter yields the following relation between R and the lifting regions $R_i := R_{\Lambda_i}(B_i, o_i)$ with $i \in \{1, 2\}$:

$$R \supseteq (1 - \mu) \bigcup_{\substack{F_1 \in \mathcal{F}^{n_1-1}(B_1) \\ z_1 \in \Lambda_1 \cap F_1}} S_{F_1,z_1}(o_1) \times \mu \bigcup_{\substack{F_2 \in \mathcal{F}^{n_2-1}(B_2) \\ z_2 \in \Lambda_2 \cap F_2}} S_{F_2,z_2}(o_2) = (1 - \mu)R_1 \times \mu R_2.$$

Consequently, using the fact that B_i has the unique-lifting property with respect to Λ_i and, by this, $R_i + \Lambda_i = \mathbb{R}^{n_i}$ for each $i \in \{1, 2\}$, we obtain

$$\begin{aligned} R + \Lambda &\supseteq (1 - \mu)R_1 \times \mu R_2 + (1 - \mu)\Lambda_1 \times \mu\Lambda_2 \\ &= (1 - \mu)(R_1 + \Lambda_1) \times \mu(R_2 + \Lambda_2) = (1 - \mu)\mathbb{R}^{n_1} \times \mu\mathbb{R}^{n_2} = \mathbb{R}^n. \end{aligned}$$

Thus, $R + \Lambda = \mathbb{R}^n$, and so B has the unique-lifting property with respect to Λ .

Case 2: $o_i \notin \text{int}(B_i)$ for some $i \in \{1, 2\}$. For each $i \in \{1, 2\}$, we have $o_i \in B_i$. Thus we can choose a sequence $(x_{i,t})_{t \in \mathbb{N}}$ of points in $\text{int}(B_i)$ converging to o_i . For every $i \in \{1, 2\}$ and $t \in \mathbb{N}$, the interior of $B_i - x_{i,t}$ contains o_i . Clearly, the set $B_i - x_{i,t}$ is maximal $(\Lambda_i - x_{i,t})$ -free and has the unique-lifting property with respect to $\Lambda_i - x_{i,t}$. Hence, we can apply the assertion obtained in Case 1. It follows that the set $(B_1 - x_{1,t}) \diamond (B_2 - x_{2,t})$ is maximal $(1 - \mu)(\Lambda_1 - x_{1,t}) \times \mu(\Lambda_2 - x_{2,t})$ -free and has the unique-lifting property with respect to this affine lattice. We introduce the vector

$$x_t := ((1 - \mu)x_{1,t}, \mu x_{2,t}),$$

and the set $B_t := (B_1 - x_{1,t}) \diamond (B_2 - x_{2,t}) + x_t$ (note that for $t \in \{1, 2\}$, there is a collision of notations, because B_1 and B_2 are already introduced; we avoid this collision by imposing the additional condition $t \geq 3$). The set B_t is maximal Λ -free and has the unique-lifting property with respect to Λ . We check that $B_t \rightarrow B = B_1 \diamond B_2$, as $t \rightarrow \infty$. Using the notation

$$\begin{aligned} B'_{1,t} &:= (B_1 - x_{1,t}) \times \{o_2\}, & B'_1 &:= B_1 \times \{o_2\}, \\ B'_{2,t} &:= \{o_1\} \times (B_2 - x_{2,t}), & B'_2 &:= \{o_1\} \times B_2, \end{aligned}$$

we obtain the following upper bounds on $\text{dist}(B_t, B)$:

$$\begin{aligned} \text{dist}(B_t, B) &= \text{dist}(\text{conv}(B'_{1,t} \cup B'_{2,t}) + x_t, \text{conv}(B'_1 \cup B'_2)) \\ &\leq \text{dist}(\text{conv}(B'_{1,t} \cup B'_{2,t}), \text{conv}(B'_1 \cup B'_2)) + \|x_t\| \\ &\leq \max \{ \text{dist}(B'_{1,t}, B'_1), \text{dist}(B'_{2,t}, B'_2) \} + \|x_t\| \end{aligned}$$

One has

$$\text{dist}(B'_{i,t}, B'_i) = \text{dist}(B_i - x_{i,t}, B_i) \leq \|x_{i,t}\|.$$

Thus, $\text{dist}(B_t, B) \leq \max \{ \|x_{1,t}\|, \|x_{2,t}\| \} + \|x_t\|$, where the right hand side of this equality converges to 0, as $t \rightarrow \infty$. We have shown that the maximal Λ -free set B_t , which has the unique-lifting property with respect to Λ , converges to the maximal Λ -free set B , as $t \rightarrow \infty$. By Theorem 4.1, we conclude that B has the unique-lifting property with respect to Λ . \square

Theorem 5.3 can be extended to a version dealing with the coproduct of $k \in \mathbb{N}$ sets.

COROLLARY 5.4. *Let $k \in \mathbb{N}$. For each $i \in \{1, \dots, k\}$, let $\mu_i > 0$, let $n_i \in \mathbb{N}$, let Λ_i be an affine lattice of rank n_i in \mathbb{R}^{n_i} and let B_i be a n_i -dimensional polytope in \mathbb{R}^{n_i} such that the origin of \mathbb{R}^{n_i} is contained in B_i . Then, for $B = \mu(B_1 \diamond \dots \diamond B_k)$ with $\mu := \mu_1 + \dots + \mu_k$ and $\Lambda := \mu_1 \Lambda_1 \times \dots \times \mu_k \Lambda_k$, the following assertions hold:*

- (a) *If B_i is Λ_i -free for each $i \in \{1, \dots, k\}$, then B is Λ -free.*
- (b) *If B_i is maximal Λ_i -free for each $i \in \{1, \dots, k\}$, then B is maximal Λ -free.*
- (c) *If B_i is maximal Λ_i -free and has the unique-lifting property with respect to Λ_i for each $i \in \{1, \dots, k\}$, then B is maximal Λ -free and has the unique-lifting property with respect to Λ .*

Proof. The assertion follows by induction, by using Theorem 5.3 in the inductive step, with the basis case $k = 1$ being trivial. \square

The following is a simple reformulation of Corollary 5.4 in a form which uses lattice-free sets rather than general Λ -free sets, where Λ is a translate of an arbitrary lattice.

COROLLARY 5.5. (Coproduct construction of various types of lattice-free sets.) *Let $k \in \mathbb{N}$. For $i \in \{1, \dots, k\}$, let $\mu_i > 0$, let $n_i \in \mathbb{N}$, let B_i be an n_i -dimensional polytope in \mathbb{R}^{n_i} and let $c_i \in B_i$. Then, for the polytope*

$$B := \frac{\mu(B_1 - c_1)}{\mu_1} \diamond \dots \diamond \frac{\mu(B_k - c_k)}{\mu_k} + (c_1, \dots, c_k)$$

with $\mu := \mu_1 + \dots + \mu_k$, the following assertions hold:

- (a) *If B_i is lattice-free for each $i \in \{1, \dots, k\}$, then B is lattice-free.*
- (b) *If B_i is maximal lattice-free for each $i \in \{1, \dots, k\}$, then B is maximal lattice-free.*
- (c) *If B_i is maximal lattice-free and has the unique-lifting property for each $i \in \{1, \dots, k\}$, then B is maximal lattice-free and has the unique-lifting property.*

Proof. For every of the three assertions the proof is based on the respective assertion of Corollary 5.4. We only prove (a), since the proofs of (b) and (c) are analogous. Assume that, for each $i \in \{1, \dots, k\}$, the set B_i is maximal lattice-free. Then $B_i - c_i$ is maximal $(\mathbb{Z}^{n_i} - c_i)$ -free. The origin of \mathbb{R}^{n_i} belongs to $B_i - c_i$. Thus, we can use Theorem 5.3 for the sets $B_i - c_i$. We obtain that the set $\mu((B_1 - c_1) \diamond \dots \diamond (B_k - c_k))$ is maximal $\mu_1(\mathbb{Z}^{n_1} - c_1) \times \dots \times \mu_k(\mathbb{Z}^{n_k} - c_k)$ -free. Using the affine transformation that sends $(x_1, \dots, x_k) \in \mathbb{R}^{n_1} \times \dots \times \mathbb{R}^{n_k}$ to $(\frac{1}{\mu_1}x_1, \dots, \frac{1}{\mu_k}x_k) + (c_1, \dots, c_k)$, we conclude that the set B is lattice-free. \square

Pyramids and double pyramids. Since pyramids and double pyramids can be described using the coproduct operation, Corollary 5.5 can be used to construct pyramids and double pyramids which have the unique-lifting property. This is presented in the following corollary.

COROLLARY 5.6. *Let B be an n -dimensional polytope in \mathbb{R}^n . Let $c \in B$, let $0 \leq \gamma < 1$ and let $0 < \mu < 1$. Then, for the polytope*

$$P := \text{conv} \left(\frac{B - \mu c}{1 - \mu} \times \{\gamma\} \cup \{c\} \times \left[\frac{(\mu - 1)\gamma}{\mu}, \frac{(\mu - 1)\gamma + 1}{\mu} \right] \right) \quad (5.16)$$

(which is a pyramid if $\gamma = 0$ and a double pyramid otherwise), the following assertions hold:

- (a) *If B is lattice-free, then P is lattice-free.*
- (b) *If B is maximal lattice-free, then P is maximal lattice-free.*
- (c) *If B has the unique-lifting property, then P has the unique-lifting property.*

Proof. A straightforward computation shows that $P = \frac{B - c}{1 - \mu} \diamond \frac{[0, 1] - \gamma}{\mu} + (c, \gamma)$, where $c \in B$ and $\gamma \in [0, 1]$. Thus, the assertion follows directly from Corollary 5.5. \square

We can also use Corollary 5.5 to provide families of simplices and cross-polytopes having the unique-lifting property.

COROLLARY 5.7. *Let $n \in \mathbb{N}$ and let $a_1, \dots, a_n > 0$ be such that $\frac{1}{a_1} + \dots + \frac{1}{a_n} = 1$. Then the following assertions hold:*

- (a) *The simplex $\text{conv}\{o, a_1 e_1, \dots, a_n e_n\}$ has the unique-lifting property.*
- (b) *The cross-polytope $\text{conv}\{\pm \frac{a_1}{2} e_1, \dots, \pm \frac{a_n}{2} e_n\} + (\frac{1}{2}, \dots, \frac{1}{2})$ has the unique-lifting property.*

Proof. For both assertions we use Corollary 5.5 with $k = n$, $B_i = [0, 1]$ and $\mu_i = \frac{1}{a_i}$ for every i . For assertion (a) we choose $c_1 = \dots = c_n = 0$, while for assertion (b) we choose $c_1 = \dots = c_n = \frac{1}{2}$. \square

Remark 5.8. As mentioned in the Introduction, without loss of generality we only deal with polytopes in this paper because maximal lattice-free polyhedra have a recession cone which is a rational linear subspace, and so the lifting region is a cylinder over the lifting region of a polytope. To use the coproduct operation on such unbounded maximal lattice-free polyhedra, one would use the operation on the polytopes - after removing the linearity spaces of the polyhedra - and then add back the direct sum of these two linear subspaces. \square

Comparison with existing results on the unique-lifting property. All known classes of maximal lattice-free polytopes with the unique-lifting property from the literature can be constructed using the formula (5.16). For simplices with the unique-lifting property:

- Setting $a_1 = a_2 = \dots = a_n = n$ gives the so-called Type 1 triangle (for $n = 2$) and its higher-dimensional generalizations that were first shown to have the unique-lifting property in [CCZ11b] and [BCK12].
- All the results on 2-partitionable simplices from Section 4 in [BCK12] can be derived using (5.16).

Of course, (5.16) can be used to create pyramids that are not simplices (for example, by creating a n -dimensional pyramid over a 2-dimensional quadrilateral with unique-lifting), and so it is a much more powerful and general construction compared to existing constructions for simplices with unique lifting. Similarly, the cross-polytope construction in Corollary 5.7 (b) for $n = 2$ gives precisely the quadrilaterals with unique-lifting property. Further, *every* maximal lattice-free polytope for $n = 2$ can be obtained using the coproduct construction (5.16).

In summary, the coproduct construction can be used to obtain every previously known maximal lattice-free polytope with the unique-lifting property, and gives a very general way to obtain new unique-lifting polytopes in higher dimensions.

The cube construction. One may wonder, in the light of previous remarks, whether for a given dimension $n \geq 2$ the coproduct construction generates all unique-lifting polytopes in \mathbb{R}^n . We saw that for $n = 2$ this is indeed the case. Below we describe a construction, which shows that for infinitely many choices of n , there exist unique-lifting polytopes which cannot be generated using the coproduct construction. First, we observe that nonsingular affine transformations of cubes of dimension at least three are not coproducts of any polytopes:

PROPOSITION 5.9. *Let $n \in \mathbb{N}$ and $n \geq 3$. Let $B \subseteq [0, 1]^n$ an image of the n -dimensional cube under a nonsingular affine transformation. Then B is not representable as $P_1 \diamond P_2$, where P_i is an n_i -dimensional polytope in \mathbb{R}^{n_i} and $n_i \in \mathbb{N}$ for each $i \in \{1, 2\}$.*

Proof. Assume the contrary, that is, $B = P_1 \diamond P_2$. For each $i \in \{1, 2\}$ choose a facet F_i of P_i with $o_i \notin F_i$, where o_i denotes the origin of \mathbb{R}^{n_i} . By Lemma 5.1, the polytopes $F = F_1 \diamond F_2$ is a facet of B , while the polytopes $F_1 \times \{o_2\}$ and $\{o_1\} \times F_2$ are faces of B . All faces of B are nonsingular affine images of cubes of dimensions at most n . A cube of dimension $k \in \mathbb{N}$ has $2k$ facets. Thus, F_i has $2(n_i - 1)$ facets. In the case that $n_i \geq 2$ for each $i \in \{1, 2\}$, the set $\mathcal{F}^{n-2}(F)$ of all facets of F is precisely the set of polytopes of the form $F_1 \diamond G_2$ and $G_1 \diamond F_2$, where G_i is a facet of F_i . Consequently, F has $2(n_1 - 1) + 2(n_2 - 1) = 2(n - 2)$ facets. On the other hand, since F is a facet of B and by this an nonsingular affine image of an $(n - 1)$ -dimensional cube, F has $2(n - 1)$ facets, which is a contradiction. We switch to the case that $n_i = 1$ for some $i \in \{1, 2\}$. Without loss of generality, let $n_2 = 1$. In this case F_2 is 0-dimensional and thus F is a pyramid with the base $F_1 \times \{o_2\}$, which is a polytope with $2(n_1 - 1)$ facets, and the apex $\{o_1\} \times F_2$. It follows that F has $2(n_1 - 1) + 1 = 2(n - 2) + 1$ facets, which is a contradiction to the fact that F has $2(n - 1)$ facets. \square

PROPOSITION 5.10. *Let $n \in \mathbb{N}$ be odd and $n \geq 3$. Let Λ be the lattice of rank n in \mathbb{R}^n given by*

$$\Lambda := \{z = (z_1, \dots, z_n) \in \mathbb{Z}^n : z_1 + \dots + z_n \text{ is even}\}.$$

Then the cube $B := [0, 2]^n$ is a maximal Λ -free polytope with the unique-lifting property with respect to Λ .

Proof. Clearly, B is Λ -free because the only integer point in the relative interior of B is $(1, \dots, 1) \in \mathbb{R}^n$. Since the dimension n is odd, this point is not in Λ . The cube B is even maximal Λ -free, because in the relative interior of every facet of B one can find a point of Λ with one component equal to 0 or 2 and the remaining components equal to 1. For the verification of the unique-lifting property we choose $f := (1, \dots, 1)$ and test whether $R_\Lambda(B, f) + \Lambda = \mathbb{R}^n$. Let D be the fundamental parallelepiped for Λ , i.e., every point in \mathbb{R}^n is uniquely representable as a point from Λ and a point from D . Define $R_\Lambda(B, f)/\Lambda = \{p \in D : (p + \Lambda) \cap R_\Lambda(B, f) \neq \emptyset\}$. Then $R_\Lambda(B, f) + \Lambda = \mathbb{R}^n$ if and only if $\text{vol}(R_\Lambda(B, f)/\Lambda) = \text{vol}(D) = \det(\Lambda) = 2$. Consider the spindle $S := S_{F,z}(f)$ associated with the facet $F := [0, 2]^{n-1} \times \{0\}$ and the lattice point $z := (1, \dots, 1, 0) \in \Lambda$, where z is the only point of Λ in the relative interior of F . Furthermore, by symmetry reasons, the parts of $R_\Lambda(B, f)$ associated to the remaining facets of B the same properties with respect to Λ as our fixed facet F . It follows, taking into account (3.4), that $\text{vol}(R_\Lambda(B, f)/\Lambda) = 2n \text{vol}(S/\Lambda)$, where $2n$ is the number of facets of B . We have $\text{relint}(F) \cap \Lambda = \{z\}$. Thus, in view of (3.3), the interior of S does not contain distinct points congruent modulo Λ (note that we apply (3.3) with Λ in place of \mathbb{Z}^n). Thus, one has $\text{vol}(S/\Lambda) = \text{vol}(S)$. Note that, since F is centrally symmetric with center at z , we see that S is a double pyramid, which can be represented by $S_{F,z}(f) = \text{conv}\{[\frac{1}{2}, \frac{3}{2}]^{n-1} \times \{\frac{1}{2}\} \cup \{z, f\}\}$, where z, f are the apexes of S and the set $[\frac{1}{2}, \frac{3}{2}]^{n-1} \times \{\frac{1}{2}\}$ is the base of S . Using the standard formula for the volume of double pyramids we obtain $\text{vol}(S) = \frac{1}{n}$. Thus, $\text{vol}(R_\Lambda(B, f)/\Lambda) = 2 = \det(\Lambda)$. It follows that B has the unique-lifting property with respect to Λ . \square

We remark that in the case of dimension $n = 3$, the polytope B discussed in the above proposition can be found in [AWW11], where the authors use the standard lattice \mathbb{Z}^3 rather than Λ . The respective polytope is written as $[o, e_1 + e_2] + [o, -e_1 + e_2] + [o, e_1 + e_2 + 2e_3]$ rather than $[0, 2]^3$. To verify the equivalence with the example of the previous proposition it suffices to check that the linear mapping sending $e_1 + e_2$ to $2e_1$, $-e_1 + e_2$ to $2e_2$ and $e_1 + e_2 + 2e_3$ to $2e_3$, which maps bijectively the mentioned polytope from [AWW11] to the cube $[0, 2]^3$, is also a bijection between \mathbb{Z}^3 and Λ . Thus, B is a maximal lattice-free set which has the unique lifting property, but is not the representable as a coproduct by Proposition 5.9.

6 Characterization of special polytopes with the unique-lifting property

Towards explicit description of polytopes with the unique-lifting property. Providing an explicit description of all n -dimensional maximal lattice-free polytopes with the unique-lifting property for $n \geq 2$ is a challenging problem: so far, only the case $n = 2$ has been settled completely. Already the case $n = 3$ seems to be highly nontrivial. In the authors'

opinion, one of the difficulties is that, in general, the set $\text{int}(R_F(B, f))$ with $f \in B \setminus F$ has complicated geometry whenever the relative interior of a facet F of an n -dimensional maximal lattice-free polytope B contains more than one integral point. It is thus interesting to analyze the somewhat more accessible special case in which $\text{relint}(F) \cap \mathbb{Z}^n$ consists of exactly one point for each facet F of B . In this section we provide some partial information on the problem described above.

We say that a closed set $S \subseteq \mathbb{R}^n$ with nonempty interior *translatively tiles* \mathbb{R}^n if \mathbb{R}^n can be represented as $\bigcup_{u \in U} (S + u)$, where $U \subseteq \mathbb{R}^n$ and $\text{int}(S + u_1) \cap \text{int}(S + u_2) = \emptyset$ for all $u_1, u_2 \in U$ with $u_1 \neq u_2$. We say that S tiles \mathbb{R}^n by its integral translations if the latter condition holds with $U = \mathbb{Z}^n$.

PROPOSITION 6.1. *Let $n \in \mathbb{N}$ and let B be a maximal lattice-free set in \mathbb{R}^n such that the relative interior of each facet of B contains exactly one integral point. Then $\text{vol}(R(B, f)) = \text{vol}(R(B, f)/\mathbb{Z}^n)$ and therefore, the following are equivalent: (a) B has the unique-lifting property, (b) $\text{vol}(R(B, f)) = 1$ for every $f \in B$ and (c) the topological closure of $\text{int}(R(B, f))$ tiles \mathbb{R}^n by its integral translations for every $f \in B$.*

Proof. Consider an arbitrary facet F of B and an arbitrary $f \in B$. By Lemma 3.2, no point of $\text{int}(R_F(B, f))$ is congruent to a point of another set $\text{int}(R_{F'}(B, f))$ modulo \mathbb{Z}^n , where F' is a facet of B with $F' \neq F$. Furthermore, taking into account the assumption on the facets of B , (3.3) implies that no two distinct points of $\text{int}(R(B, f))$ are congruent modulo \mathbb{Z}^n . Hence $\text{vol}(R(B, f)) = \text{vol}(R(B, f)/\mathbb{Z}^n) \leq 1$. The latter implies the assertions (a), (b) and (c). \square

Thus, in our special situation, for verifying whether B has the unique-lifting property, it suffices to compute the volume of the entire lifting region $R(B, f)$ rather than the set $R(B, f)/\mathbb{Z}^n$, which is a simplification. Nevertheless, since $R(B, f)$ is still quite a complicated set (e.g., not necessarily a convex one), checking $\text{vol}(R(B, f)) = 1$ is not an easy task.

Special pyramids with the unique-lifting property. We analyze and partially characterize pyramids with the unique-lifting property. We shall use the following theorem, which is proved in the appendix.

THEOREM 6.2. (McMullen [McM13].) *Let $S \subseteq \mathbb{R}^n$ be an n -dimensional spindle that translatively tiles space. Then S is the image of the n -dimensional hypercube under an invertible affine transformation.*

Theorem 6.2 can be used to prove the following.

THEOREM 6.3. *Let $n \in \mathbb{N}$ and let $B \subseteq \mathbb{R}^n$ be a maximal lattice-free polytope with the unique-lifting property such that B is a pyramid whose base contains exactly one integral point in the relative interior. Then B is a simplex.*

Proof. Let f be the apex and F be the base of B and let z be the unique integral point in $\text{relint}(F)$. Since B has the unique-lifting property, by Proposition 6.1(c), the topological closure of $\text{int}(R(B, f))$ tiles \mathbb{R}^n by its integral translations. The topological closure of $\text{int}(R(B, f))$ is $S = S_{F,z}(f)$, since $S_{F,z}(f)$ is the only full-dimensional spindle involved in the definition of $R(B, f)$. Thus, S tiles \mathbb{R}^n by translations. By Theorem 6.2, S is an image of a cube under an invertible affine transformation. In particular, the tangent cone at the apex f is a simple cone. Therefore, B is a simplex. \square

The following theorem is proved in [BCK12].

THEOREM 6.4. *Let $n \in \mathbb{N}$ and let B be a maximal lattice-free simplex in \mathbb{R}^n such that each facet of B has exactly one integer point in its relative interior. Then B has the unique-lifting property if and only if B is an affine unimodular transformation of $\text{conv}(\{o, ne_1, \dots, ne_n\})$.*

We can now generalize this result to pyramids.

THEOREM 6.5. *Let $n \in \mathbb{N}$ and let B be a maximal lattice-free pyramid in \mathbb{R}^n such that every facet of P contains exactly one integer point in its relative interior. Then B has the unique-lifting property if and only if B is an affine unimodular transformation of $\text{conv}(\{o, ne_1, \dots, ne_n\})$.*

Proof. Sufficiency follows from Corollary 5.7.(a). For showing the necessity we assume that B has the unique-lifting property. By Theorem 6.3, B is a simplex. Thus, Theorem 6.4 can be applied and the necessity follows immediately. \square

Acknowledgements

We are grateful to Peter McMullen for providing an outline of the proof of Theorem B.3, which is implemented in Section B of the appendix. We also would like to thank Martina Zähle for pointing to [KP08].

References

- [Ave12] G. Averkov, *On finitely generated closures in the theory of cutting planes*, Discrete Optim. **9** (2012), no. 4, 209–215.
- [Ave13] ———, *A proof of Lovász’s theorem on maximal lattice-free sets*, Beitr. Algebra Geom. **54** (2013), no. 1, 105–109.
- [AWW11] G. Averkov, Ch. Wagner, and R. Weismantel, *Maximal lattice-free polyhedra: finiteness and an explicit description in dimension three*, Math. Oper. Res. **36** (2011), no. 4, 721–742.
- [BCC⁺13] A. Basu, M. Campêlo, M. Conforti, G. Cornuéjols, and G. Zambelli, *Unique lifting of integer variables in minimal inequalities*, Math. Program. **141** (2013), no. 1-2, Ser. A, 561–576.
- [BCCZ10] A. Basu, M. Conforti, G. Cornuéjols, and G. Zambelli, *Maximal lattice-free convex sets in linear subspaces*, Math. Oper. Res. **35** (2010), no. 3, 704–720.
- [BCK12] A. Basu, G. Cornuéjols, and M. Köppe, *Unique minimal liftings for simplicial polytopes*, Math. Oper. Res. **37** (2012), no. 2, 346–355.
- [BHKM13] A. Basu, R. Hildebrand, M. Köppe, and M. Molinaro, *A $(k+1)$ -slope theorem for the k -dimensional infinite group relaxation*, SIAM Journal on Optimization **23** (2013), no. 2, 1021–1040.

- [CCZ11a] M. Conforti, G. Cornuéjols, and G. Zambelli, *Corner polyhedra and intersection cuts*, Surveys in Operations Research and Management Science **16** (2011), 105–120.
- [CCZ11b] M. Conforti, G. Cornuéjols, and G. Zambelli, *A geometric perspective on lifting*, Oper. Res. **59** (2011), no. 3, 569–577.
- [Doi73] J.-P. Doignon, *Convexity in crystallographic lattices*, Journal of Geometry **3** (1973), 71–85.
- [DW10a] S. S. Dey and L. A. Wolsey, *Composite lifting of group inequalities and an application to two-row mixing inequalities*, Discrete Optim. **7** (2010), no. 4, 256–268.
- [DW10b] ———, *Two row mixed-integer cuts via lifting*, Math. Program. **124** (2010), no. 1-2, Ser. B, 143–174.
- [GJ72a] R. E. Gomory and E. L. Johnson, *Some continuous functions related to corner polyhedra*, Math. Programming **3** (1972), 23–85.
- [GJ72b] ———, *Some continuous functions related to corner polyhedra. II*, Math. Programming **3** (1972), 359–389.
- [Gom60] R. E. Gomory, *An algorithm for the mixed integer problem*, Tech. report, DTIC Document, 1960.
- [Gom69] ———, *Some polyhedra related to combinatorial problems*, Linear Algebra and Appl. **2** (1969), 451–558.
- [Gru07] P. M. Gruber, *Convex and Discrete Geometry*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 336, Springer, Berlin, 2007.
- [HRGZ97] M. Henk, J. Richter-Gebert, and G. M. Ziegler, *Basic properties of convex polytopes*, Handbook of discrete and computational geometry, CRC Press Ser. Discrete Math. Appl., CRC, Boca Raton, FL, 1997, pp. 243–270.
- [Joh74] E. L. Johnson, *On the group problem for mixed integer programming*, Mathematical Programming Study **2** (1974), 137–179.
- [KP08] S. G. Krantz and H. R. Parks, *Geometric Integration Theory*, Cornerstones, Birkhäuser Boston, Inc., Boston, MA, 2008.
- [Len83] H. W. Lenstra, Jr., *Integer programming with a fixed number of variables*, Math. Oper. Res. **8** (1983), no. 4, 538–548.
- [Lov89] L. Lovász, *Geometry of numbers and integer programming*, Mathematical programming (Tokyo, 1988), Math. Appl. (Japanese Ser.), vol. 6, SCIPRESS, Tokyo, 1989, pp. 177–201.
- [McM70] P. McMullen, *Polytopes with centrally symmetric faces*, Israel J. Math. **8** (1970), 194–196.

- [McM76] ———, *Polytopes with centrally symmetric facets*, Israel J. Math. **23** (1976), no. 3-4, 337–338.
- [McM13] ———, *Personal communication*, 2013.
- [Roc70] R. T. Rockafellar, *Convex Analysis*, Princeton Mathematical Series, No. 28, Princeton University Press, Princeton, N.J., 1970.
- [Sca77] H. Scarf, *An observation on the structure of production sets with indivisibilities*, Proceedings of the National Academy of Sciences **74** (1977), 3637–3641.
- [Sch93] R. Schneider, *Convex bodies: the Brunn-Minkowski theory*, Encyclopedia of Mathematics and its Applications, vol. 44, Cambridge University Press, Cambridge, 1993.
- [Zie95] G. M. Ziegler, *Lectures on Polytopes*, Graduate Texts in Mathematics, vol. 152, Springer-Verlag, New York, 1995.

A Proofs of Propositions 1.1 and 1.2

Proof of Proposition 1.1. First we show that the infimum in (1.7) is attained.

Consider the case of a bounded B . For a sufficiently large $N \in \mathbb{N}$ the Euclidean ball of radius N centered at o contains B . It follows that $\phi_{B-f}(r) \geq \frac{1}{N}\|r\|$ for every $r \in \mathbb{R}^n$, where $\|\cdot\|$ is the Euclidean norm. Consequently, $\phi_{B-f}(r+w) \geq \frac{1}{N}\|r+w\| \geq \frac{1}{N}(\|w\| - \|r\|) > \phi_{B-f}(r)$ for $r \in \mathbb{R}^n$ and $w \in \mathbb{Z}^n$ whenever w fulfills $\|w\| > (N\phi_{B-f}(r) + \|r\|)$. It follows that $\inf_{w \in \mathbb{Z}^n} \phi_{B-f}(r+w)$ is attained for some of finitely many vectors $w \in \mathbb{Z}^n$ satisfying $\|w\| \leq (N\phi_{B-f}(r) + \|r\|)$.

Let us switch to the case that B is unbounded. It is known that the recession cone of B is a linear space spanned by rational vectors; see [Lov89], [BCCZ10] and [Ave13]. Up to appropriate unimodular transformations, we can assume that B has the form $B = B' \times \mathbb{R}^k$, where $k \in \{1, \dots, n-1\}$ and B' is a bounded maximal lattice-free set in \mathbb{R}^{n-k} . We denote by $\phi_{B'-f}$ the gauge-function of B' ; it is well known that $\phi_{B-f}((r', r'')) = \phi_{B'-f}(r')$ for all $(r', r'') \in \mathbb{R}^{n-k} \times \mathbb{R}^k$. Thus, it suffices to apply the assertion of the bounded case to B' to get the assertion for an unbounded B .

It remains to prove the assertion on polynomial-time computability. Assume that $n \in \mathbb{N}$ is fixed and that f and a_i ($i \in I$) are rational vectors, whose components are given as the input in standard binary encoding. We have

$$\begin{aligned}
\phi_{B-f}^*(r) &:= \min_{w \in \mathbb{Z}^n} \phi_{B-f}(r+w) \\
&= \min_{w \in \mathbb{Z}^n} \max_{i \in I} a_i \cdot (r+w) \\
&= \min \{ \rho \geq 0 : \rho \geq a_i(r+w) \ \forall i \in I, \ w \in \mathbb{Z}^n \}.
\end{aligned} \tag{A.1}$$

Expression (A.1) defines a mixed-integer linear program with rational coefficients with n integer variables (the components of w) and one real variable (the value ρ). Since n is fixed, Lenstra's algorithm [Len83] can be used to determine (A.1) in polynomial time. \square

Proof of Proposition 1.2. It was established in [BCC⁺13] that for every $r \in \mathbb{R}^n$ such that $r + f \in R(f, B)$, $\pi(r) = \phi_{B-f}(r)$ for every minimal lifting π of ϕ_{B-f} . Moreover, it is not difficult to see that every minimal lifting is periodic with respect to \mathbb{Z}^n , i.e., $\pi(r) = \pi(r + w)$ for every $r \in \mathbb{R}^n$ and $w \in \mathbb{Z}^n$. If ϕ_{B-f} has a unique lifting, then $R(f, B) + \mathbb{Z}^n = \mathbb{R}^n$. Therefore, for any r , there exists $w \in \mathbb{Z}^n$ such that $r + w + f \in R(f, B)$ and thus $\pi(r) = \pi(r + w) = \phi_{B-f}(r + w) \geq \phi_{B-f}^*(r)$ for every minimal lifting π , thus establishing that ϕ_{B-f}^* is a minimal lifting.

Suppose ϕ_{B-f} does not have a unique minimal lifting. This implies there are at least two distinct minimal liftings and so there must exist a minimal lifting π that is different from the lifting ϕ_{B-f}^* . However, we show below that $\pi \leq \phi_{B-f}^*$. Thus, ϕ_{B-f}^* is not a minimal lifting.

To show that $\pi \leq \phi_{B-f}^*$, consider any $r \in \mathbb{R}^n$. It is well-known that $\pi \leq \phi_{B-f}$ because π is a minimal lifting. By Theorem 1.1, there exists $w \in \mathbb{Z}^n$ such that $\phi_{B-f}^*(r) = \phi_{B-f}(r + w)$. By the \mathbb{Z}^n -periodicity of π , we have $\pi(r) = \pi(r + w) \leq \phi_{B-f}(r + w) = \phi_{B-f}^*(r)$. \square

B Proof of Theorem 6.2

Let $P \subseteq \mathbb{R}^n$ be an n -dimensional centrally symmetric polytope with centrally symmetric facets. Let G be any $(n - 2)$ -dimensional face of P . The *belt* corresponding to G is the set of all facets which contain a translate of G or $-G$. Observe that every centrally symmetric polytope P with centrally symmetric facets has belts of even size greater than or equal to 4.

A *zonotope* is a polytope given by a finite set of vectors $V = \{v_1, \dots, v_k\} \subseteq \mathbb{R}^n$ in the following way: $Z(V) := \{\lambda_1 v_1 + \dots + \lambda_k v_k : -1 \leq \lambda_i \leq 1 \ \forall i = 1, \dots, k\}$. We recall that $F(P, u)$ denotes the face of points in P maximizing the linear function $x \mapsto u \cdot x$. The following simple lemma is well-known.

LEMMA B.1. *Let $n \in \mathbb{N}$. Let V be a nonempty finite subset of \mathbb{R}^n and let $u \in \mathbb{R}^n$. Then the face $F(Z(V), u)$ of the zonotope $Z(V)$ coincides, up to a translation, with the zonotope $Z(\{v \in V : u \cdot v = 0\})$.*

Proof. By the Minkowski additivity of the functional $F(\cdot, u)$, defined by (4.1), we get $F(Z(V), u) = \sum_{v \in V} F([-v, v], u)$. It is straightforward to verify that for every $v \in V$ one has

$$F([-v, v], u) := \begin{cases} \{-v\} & \text{if } u \cdot v < 0, \\ [-v, v] & \text{if } u \cdot v = 0, \\ \{v\} & \text{if } u \cdot v > 0. \end{cases}$$

Putting these observations together, we have the assertion. \square

The latter lemma shows that every face of a zonotope is a zonotope (and, thus, centrally symmetric). The following lemma deals with belts of zonotopes. Each belt of the cube $[-1, 1]^n$ consists of exactly four facets. The following theorem shows that the latter property essentially characterizes cubes within all zonotopes.

THEOREM B.2. *Let $n \in \mathbb{N}$, $n \geq 3$. Let V be a finite set linearly spanning \mathbb{R}^n and such that each belt of the n -dimensional zonotope $Z(V)$ consists of exactly four facets. Then $Z(V)$ is the image of the n -dimensional cube $[-1, 1]^n$ under a invertible linear transformation.*

Proof. Choose a basis b_1, \dots, b_n of \mathbb{R}^n consisting of vectors in V . It suffices to show that every vector of V is parallel to some vector of $\{b_1, \dots, b_n\}$. After a change of coordinates in \mathbb{R}^n we can assume that b_1, \dots, b_n is the standard basis e_1, \dots, e_n .

Assume to the contrary, that there exists a vector $a = (\alpha_1, \alpha_2, \dots, \alpha_n) \in V$ which is not parallel to any vector of the basis e_1, \dots, e_n . Thus, at least two of its components $\alpha_1, \dots, \alpha_n$ are nonzero. Without loss of generality let $\alpha_1 \neq 0$ and $\alpha_2 \neq 0$. Let $W := V \cap (\{0\}^2 \times \mathbb{R}^{n-2})$. We have $e_3, \dots, e_n \in W$ and $e_1, e_2, a \in V \setminus W$. Choose a nonzero vector $u' = \mathbb{R}^2 \times \{0\}^{n-2}$ such that u' is not orthogonal to any vector from $V \setminus W$ (e.g., one can choose $u' = (1, \varepsilon, 0, \dots, 0)$, where $\varepsilon > 0$ is small). By Lemma B.1, the face $G := F(Z(V), u')$ is a translation of $Z(W)$. By the choice of W , the zonotope $Z(W)$ is $(n-2)$ -dimensional. We analyze the belt of $Z(V)$ determined by the $(n-2)$ -dimensional face G .

We shall construct a number of facets $F(Z(V), u)$ with $u \in \mathbb{R}^2 \times \{0\}^{n-2}$ belonging to the belt generated by G . In view of Lemma B.1, for $u = e_1$ the face $F(Z(V), u)$ contains a translation of $Z(\{e_2\} \cup W)$. Similarly, for $u = e_2$ the face $F(Z(V), u)$ contains a translation of $Z(\{e_1\} \cup W)$. For a nonzero vector $u \in \mathbb{R}^2 \times \{0\}^{n-2}$ orthogonal to a (say, for $u = (-\alpha_2, \alpha_1, 0, \dots, 0)$) the face $F(Z(V), u)$ contains a translation of $Z(\{a\} \cup W)$. Since the zonotopes $Z(\{e_1\} \cup W)$, $Z(\{e_2\} \cup W)$ and $Z(\{a\} \cup W)$ are $(n-1)$ -dimensional, we see that for all three choices of u above, the face $F(Z(V), u)$ is actually a facet. The latter shows that the six distinct facets $F(Z(V), u)$ with $u \in \{\pm e_1, \pm e_2, \pm(-\alpha_2, \alpha_1, 0, \dots, 0)\}$ belong to the belt generated by G . The latter is a contradiction to the assumptions on $Z(V)$. \square

THEOREM B.3. (McMullen [McM13].) *Let $n \in \mathbb{N}$, $n \geq 3$, and let $S \subseteq \mathbb{R}^n$ be an n -dimensional spindle with centrally symmetric facets. Then S is the image of the n -dimensional hypercube under an invertible affine transformation.*

Proof. Since all facets of S are centrally symmetric, by the Alexandrov-Shephard theorem (see [McM76] for a short proof), the polytope S itself is also centrally symmetric. Without loss of generality, we assume that S is symmetric in the origin. Let a and $-a$ be the apexes of the spindle S .

We first show that every belt of S is of length 4. Let G be an arbitrary $(n-2)$ -dimensional face of S and consider the belt of S associated with G . Since S is centrally symmetric, each belt is even length, i.e., of length $2k$ where $k \geq 2$. There are k facets F_1, \dots, F_k involved in this belt that contain a ; the remaining k facets contain $-a$. We project S onto the two-dimensional space perpendicular to G to get a polygon P . The facets F_1, \dots, F_k are all projected onto k distinct edges of the polygon P . Moreover, observe the projection of a is contained in all these edges. Since P is two-dimensional, intersection of more than three edges of P is empty. Hence $k \leq 2$ and since we also have $k \geq 2$, we get $k = 2$.

We next show that all faces of S are centrally symmetric. To do this, we first show that every $n-2$ -dimensional face G is centrally symmetric (for $n = 3$ this is clear). For $i \in \{1, 2\}$, by c_i we denote the center of symmetry of F_i . Then G has the form $F_i \cap F_j$ or $(-F_i) \cap (-F_j)$ or $F_i \cap (-F_j)$ with appropriate i, j satisfying $\{i, j\} = \{1, 2\}$. Consider the case $G = F_i \cap F_j = F_1 \cap F_2$. The symmetry of F_1 implies that $2c_1 - G$ (the reflection of G with respect to c_1) is a face of F_1 . Since $a \in G$, the face $2c_1 - G$ does not contain a . The face G is contained in exactly two facets of S , both belonging to the belt $\{F_1, F_2, -F_1, -F_2\}$ generated by G . The facet $-F_1$ cannot contain $2c_1 - G$, because $-F_1$ is opposite to F_1 and thus does not share any nonempty face with F_1 . The facet F_2 of S cannot contain $2c_1 - G$, because F_2

contains a , while $2c_1 - G$ does not contain a . It follows that the facet $-F_2$ contains $2c_1 - G$. Then the reflection $-2c_2 - (2c_1 - G)$ of $2c_1 - G$ with respect to the center $-c_2$ of $-F_2$ is a facet of $-F_2$. On the other hand, the reflection $-G$ of G with respect to the center o of S is a face of S which does not contain a . Hence $-G$ is a facet of $-F_2$. We have shown that $2c_1 - G$, $-2c_2 - 2c_1 + G$ and $-G$ are facets of $-F_2$. Since all these facets of F_2 are parallel, two of them must coincide. We cannot have $2c_1 - G = -G$, since this would imply $c_1 = o$ and, by this, $\text{relint}(F_1) \cap \text{int}(S) \neq \emptyset$, which is a contradiction. Consequently, $2c_1 - G = -2c_2 - 2c_1 + G$ or $-2c_2 - 2c_1 + G = -G$, where each of the two equalities implies that G is centrally symmetric. The case $G = (-F_i) \cap (-F_j)$ is completely analogous to the case $G = F_i \cap F_j$.

Let us switch to the case $G = F_i \cap (-F_j)$ with $\{i, j\} = \{1, 2\}$. Without loss of generality, let $G = F_1 \cap (-F_2)$. The face G of S contains neither a nor $-a$. The same also holds for the face $-G$ of S . The reflection $2c_1 - G$ of G with respect to the center c_1 of F_1 is a facet of F_1 . Then $2c_1 - G$ is not a facet of $-F_1$, because $2c_1 - G$ is a facet of F_1 , while F_1 and $-F_1$ are opposite facets of S . Thus, $2c_1 - G$ is a facet of F_2 or $-F_2$. If $2c_1 - G$ is a facet of F_2 , then also $2c_2 - (2c_1 - G)$ is a facet of F_2 . It follows that $2c_1 - G$, $2c_2 - 2c_1 + G$ and $-G$ are facets of F_2 . Again, since they are all parallel, two of them must coincide. Coincidence of $2c_1 - G$ and $-G$ implies $c_1 = o$ and yields a contradiction. Coincidence of any two other of these three facets of F_2 implies that G is centrally symmetric. In the case that $2c_1 - G$ is a facet of $-F_2$, we get that $-2c_2 - (2c_1 - G)$ is a facet of $-F_2$. Thus, G , $-2c_2 - G$ and $-2c_2 - 2c_1 + G$ are facets of $-F_2$. Coincidence of G and $-2c_2 - 2c_1 + G$ implies $c_1 = c_2$, yielding $\text{relint}(F_1) \cap \text{relint}(F_2) \neq \emptyset$, which is a contradiction. Coincidence of any other of these three facets of S implies that G is centrally symmetric.

It follows that every $(n - 2)$ -dimensional face of S is centrally symmetric. Therefore, by a theorem of McMullen [McM70], in the case $n \geq 4$, every face of S is centrally symmetric (in the case $n = 3$ this is clear from the assumptions). Consequently, all 2-dimensional faces of S are centrally symmetric and, by this, S is a zonotope; see, for example, [Sch93, Theorem 3.5.1]. Since S is a zonotope whose belts are length 4, by Lemma B.1, S is the image of the n -dimensional hypercube under an invertible affine transformation. \square

Theorem B.3 was communicated to us by Peter McMullen via personal email. We include a complete proof here as the result does not appear explicitly in the literature. The above proof is based on a proof sketch by Prof. McMullen.

We now state the celebrated Venkov-Alexandrov-McMullen theorem on translative tilings.

THEOREM B.4. (Venkov-Alexandrov-McMullen; see [Gru07, Theorem 32.2].) *Let P be a compact convex set with nonempty interior that translatively tiles \mathbb{R}^n . Then the following assertions hold:*

- (a) *P is a centrally symmetric polytope.*
- (b) *All facets of P are centrally symmetric.*
- (c) *Every belt of P is either length 4 or 6.*

Proof of Theorem 6.2. We only need to consider the case $n \geq 3$. The assertion follows directly from Theorem B.4 (assertions (a) and (b)) and Theorem B.3. \square